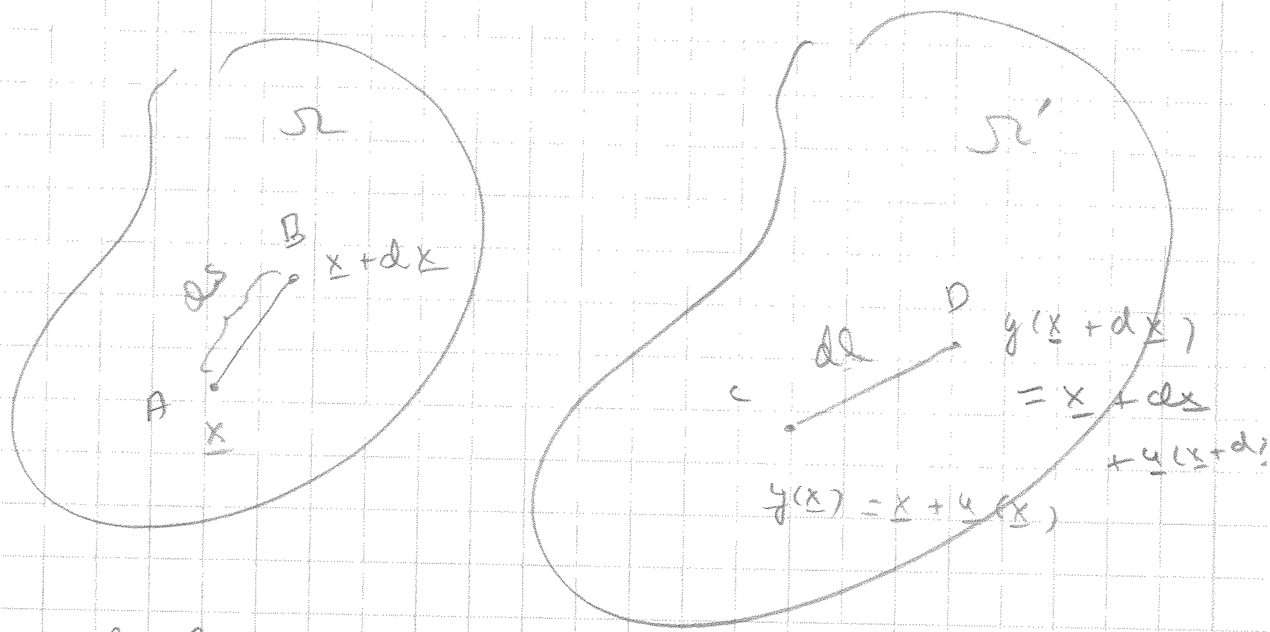
The strain tensor



We have

$$d\underline{y} = \nabla \underline{y}(\underline{x})\, d\underline{x},$$

where $\nabla \underline{y}(\underline{x})$ is the deformation gradient.

With index notation:

$$dy_i = y_{i,j}\, dx_j .$$

Therefore

$$dl^2 = |\overline{CD}|^2 = dy_i\, dy_i$$

$$= y_{i,k}\, dx_k \cdot y_{i,l}\, dx_l$$

$$= y_{i,k}\, y_{i,l}\, dx_k\, dx_l .$$

On the other hand

$$(ds)^2 = (\overline{AB})^2 = dx_j \, dx_j = \delta_{kl} \, dx_k \, dx_l$$

Hence,

$$(dl)^2 - (ds)^2 = (y_{i,k} \, y_{i,l} - \delta_{kl}) \, dx_k \, dx_l .$$

With matrices:

$$(dl)^2 - (ds)^2 = d\underline{x}^T \, (\nabla \underline{y}^T \, \nabla \underline{y} - \underline{I}) \, d\underline{x} .$$

Substituting the expression for the displacement

$$\underline{u}(\underline{x}) = \underline{y}(\underline{x}) - \underline{x} \quad \Leftrightarrow \quad \underline{y}(\underline{x}) = \underline{x} + \underline{u}(\underline{x})$$

or

$$y_i = x_i + u_i$$

we obtain

$$y_{i,k} = u_{i,k} + \delta_{ik} ,$$

$$\nabla \underline{y} = \nabla \underline{u} + \underline{I} .$$

and

$$y_{i,k}\, y_{i,l} - \delta_{kl}$$

$$= (u_{i,k} + \delta_{ik})(u_{i,l} + \delta_{il}) - \delta_{kl}$$

$$= u_{i,k}\, u_{i,l} + u_{k,l} + u_{l,k} + \delta_{ik}\delta_{il} - \delta_{kl}$$

$$= u_{i,k}\, u_{i,l} + u_{k,l} + u_{l,k}.$$

We denote

$$2\,\varepsilon_{ij} = (u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j})$$

and it holds

$$(dl)^2 - (ds)^2 = 2\,\varepsilon_{ij}\, dx_i\, dx_j .$$

With matrices: $\underline{y} = \underline{x} + \underline{u} \Rightarrow$

$$\nabla \underline{y} = \underline{I} + \nabla \underline{u}.$$

and

$$(dl)^2 - (ds)^2 = d\underline{x}^T (\underline{I} + \nabla \underline{u}^T)(\underline{I} + \nabla \underline{u})\, d\underline{x}$$

$$= d\underline{x}^T (\nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u})\, d\underline{x},$$

For small displacement gradients $\nabla u^T \nabla u \approx \underline{0}$,
and

$$\varepsilon_{ij} \approx e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

In the sequel we will use the notation ε_{ij} instead of e_{ij}.

What is the physical interpretation of ε_{ij}?

We have

$$dy_i dy_i = 2 \varepsilon_{kl} dy_k dy_l .$$

In general, if we consider $d\underline{x}$, $d\underline{x}'$ that are mapped to $d\underline{y}$ and $d\underline{y}'$ we have

$$dy_i dy_i' = 2 \varepsilon_{kl} dx_k dx_l'$$

Let $\underline{dx} = \underline{dx}' = (dx_1, 0, 0)$

Then

$(dl)^2 - (ds)^2$

$$= 2\varepsilon_{kl}\, dx_k\, dx_l = 2\varepsilon_{11}(dx_1)^2$$

Hence,

$$(dl)^2 = (2\varepsilon_{11} + 1)|dx_1|^2$$

and

$$\frac{|dl| - |dx_1|}{|dx_1|} = (1 + 2\varepsilon_{11})^{1/2} - 1 .$$

Linearizing, we get

$$\frac{|dl| - |dx_1|}{|dx_1|} \approx \varepsilon_{11} .$$

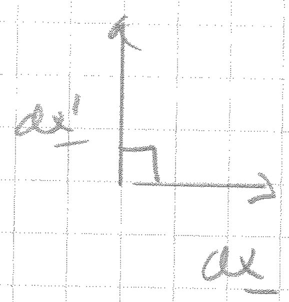
Next, ε_{12} ..

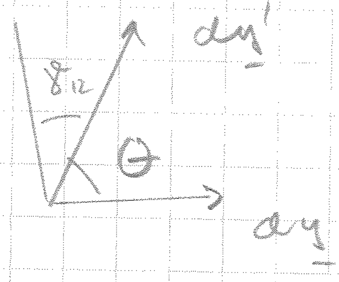
Let $d\underline{x} = (dx_1, 0, 0)$

and $d\underline{x}' = (0, dx_2', 0)$

Then

$$d\underline{y} \cdot d\underline{y}' = 2 \varepsilon_{12} \, dx_1 \, dx_2'$$





We have

$$\cos\theta = \frac{d\underline{y} \cdot d\underline{y}'}{|d\underline{y}|\,|d\underline{y}'|}$$

$$= \frac{2\varepsilon_{12}}{(1+2\varepsilon_{11})^{1/2}(1+2\varepsilon_{22})^{1/2}} .$$

Let $\gamma_{12} = \pi/2 - \theta$ be the shear angle. It hold

$$\sin\gamma_{12} = \cos\theta = \frac{2\varepsilon_{12}}{(1+2\varepsilon_{11})^{1/2}(1+2\varepsilon_{22})^{1/2}}$$

$$\gamma_{12} \simeq \sin\gamma_{12} \approx 2\varepsilon_{12} .$$

The infinitesimal rotation tensor

We have

$$du_i = u_{i,j}\, dx_j$$

We split

$$u_{i,j} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) - \tfrac{1}{2}(u_{i,j} - u_{j,i})$$

$$= e_{ij} + \omega_{ij}, \quad \text{with}$$

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}).$$

Hence, the tensor ω_{ij} is antisymmetric: $\omega_{ij} = -\omega_{ji}$.

Note also by writing

$$\underline{\omega} = (-\omega_{23}, \omega_{13}, -\omega_{12})$$

it holds

$$\underline{\omega}\, \underline{s} = \omega_{ij}\, s_j = \underline{\omega} \times \underline{s}.$$

Since

$$\underline{\omega} \times \underline{s} = (\omega_{13} s_3 + \omega_{12} s_2, \; \omega_{23} s_3 - \omega_{12} s_1, \; -\omega_{23} s_2 - \omega_{13} s_1)$$

$\square$

HOOKE'S LAW

We assume that the stress depends linearly on the strain.

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} e_{kl}$$

or

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

or

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\, \underline{\underline{e}}(\underline{u}).$$

Number of coefficients :

but

$$C_{ijkl} = C_{jikl}$$

and

$$C_{ijkl} = C_{ijlk},$$

From energy considerations one can prove that

$$C_{ijkl} = C_{klij}$$

This reduces the independent constants to 21.

We define an <u>isotropic material</u> a material for which the coeffients C_{ijkl} are independent of the chosen coordinate system. It holds

<u>Theorem.</u> For an isotropic material there is two positive constants (Lamé) μ, λ such that

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} .$$

<u>Remark.</u> $\mu = G$ = the shear modulus = liukumoduli.

<u>Proof</u>. We first prove that for an isotropic material the principal directions for the stress and strain coincide.

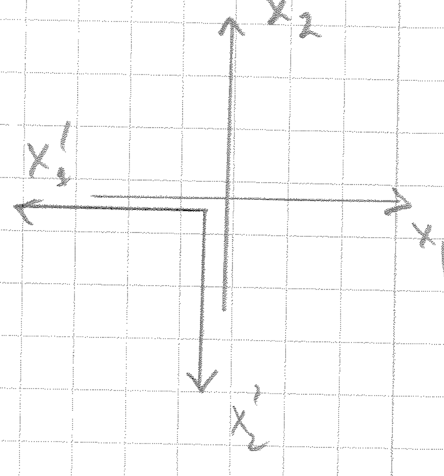
We choose the coordinate system according to the principal stresses. Hence,

$$\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 .$$

We next prove that (e.g.)

$\tau_{23} = 0$,

We choose a new coordinate system by rotating $180°$ around the x_3-axis



The matrix of the coordinate transform is

$$\underline{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We get

$$\underline{\underline{\varepsilon}}' = \underline{A}\,\underline{\varepsilon}\,\underline{A}^T = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix},$$

i.e. $\varepsilon_{ii}' = \varepsilon_{ii}$ and

$\varepsilon_{ij}' = 0 \quad i \neq j.$

For the stress tensor we have

$$\underline{\underline{\sigma}}' = \underline{\underline{A}}\ \underline{\underline{\sigma}}\ \underline{\underline{A}}^T = \begin{pmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{12} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{13} & -\sigma_{23} & \sigma_{33} \end{pmatrix},$$

Hence, $\sigma'_{23} = -\sigma_{23}$.

From the assumption of isotropy we obtain

$$\sigma'_{23} = A\varepsilon'_{11} + B\varepsilon'_{22} + C\varepsilon'_{33}$$
$$= A\varepsilon_{11} + B\varepsilon_{22} + C\varepsilon_{33}$$
$$= \sigma_{23}.$$

We thus have $\sigma'_{23} = -\sigma'_{23} \Rightarrow$

$$\sigma'_{23} = 0$$

In the same way we obtain

$$\sigma'_{13} = 0 \quad \text{and} \quad \sigma'_{12} = 0.$$

Next, we consider (e.g.) the component σ_{11}. We have

$$\sigma_{11} = a\,\varepsilon_{11} + b\,\varepsilon_{22} + c\,\varepsilon_{33}.$$

We rotate the coordinate system

$90°$, around the x_1-axis

so that $x_1' = x_1$, $x_2' = x_3$, $x_3' = -x_2$.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

we obtain

$$\underline{\underline{\sigma}}' = \underline{\underline{A}}\,\underline{\underline{\sigma}}\,\underline{\underline{A}} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{33} & 0 \\ 0 & 0 & \sigma_{22} \end{pmatrix}$$

Similarly, $\varepsilon_{11}' = \varepsilon_{11}$, $\varepsilon_{22}' = \varepsilon_{33}$

and $\varepsilon_{33}' = \varepsilon_{22}$.

Hence, we have

$$\sigma_{11}' = a\,\varepsilon_{11}' + b\,\varepsilon_{22}' + c\,\varepsilon_{33}'$$

$$= a\,\varepsilon_{11} + b\,\varepsilon_{33} + c\,\varepsilon_{22}.$$

On the other hand

$$\sigma_{11}' = \sigma_{11} = a\,\varepsilon_{11} + b\,\varepsilon_{22} + c\,\varepsilon_{33},$$

which implies

$$b = c.$$

We write

$$\sigma_{11} = a\,\varepsilon_{11} + b\,(\varepsilon_{22} + \varepsilon_{33})$$

$$= \underbrace{(a-b)}_{:= 2\mu}\,\varepsilon_{11} + \underbrace{b}_{:= \lambda}\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

In the same way we conclude that

$$\sigma_{22} = 2\mu\,\varepsilon_{22} + \lambda\,\varepsilon_{kk},$$

$$\sigma_{33} = 2\mu\,\varepsilon_{33} + \lambda\,\varepsilon_{kk}.$$

In the primical axis system we thus have

$$\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{\varepsilon}} + \lambda\,\mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{I}}$$

and the final result is obtained by transforming to a general coordinate system:

$$\underline{\underline{\sigma}}' = \underline{\underline{A}}\,\underline{\underline{\sigma}}\,\underline{\underline{A}}^T = 2\mu\,\underline{\underline{A}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{A}}^T$$

$$+ \lambda\,\mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{A}}\,\underline{\underline{A}}^T$$

$$= 2\mu\,\underline{\underline{A}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{A}}^T + \lambda\,\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{A}}^T)\,\underline{\underline{I}}.$$

$$= 2\mu\,\underline{\underline{\varepsilon}}' + \lambda\,\mathrm{tr}(\underline{\underline{\varepsilon}}')\,\underline{\underline{I}} \quad \square.$$

There are quite a few ways of choosing the two material constants, e.g. the following are used:

The <u>bulk modulus</u>:

$$K = k = \lambda + \frac{2}{3}\mu.$$

<u>Young's modulus</u> (kimmokerroin)

$$E = \frac{(3\lambda + 2\mu)\mu}{\mu + \lambda}$$

The <u>Poisson ratio</u>

$$\nu = \frac{\lambda}{2(\lambda+\mu)}.$$

In engineering one usually use E and ν. With these it holds

$$\underline{\underline{\varepsilon}} = \frac{1}{E}\left[(1+\nu)\underline{\underline{\sigma}} - \nu \operatorname{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}\right].$$

Conservation of energy

Uniqueness of the dynamic sol.

The equations of motion for the linear elasticity problem are

$$\rho \ddot{u}_i = (2\mu \varepsilon_{ij}(\underline{u}) + \lambda \varepsilon_{kk}(\underline{u})\delta_{ij})_{,j} + f_i$$

where we have substituted the constitutive law

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk}\delta_{ij}$$

into the eqs of motion

$$\rho \ddot{u}_{i} = \sigma_{ij,j} + f_i .$$

Since $\varepsilon_{ij} = \varepsilon_{ji}$ the rotational equations of motion $\sigma_{ij} = \sigma_{ji}$ are satisfied.

Remark. We can develop the right hand side:

$$(2\mu \varepsilon_{ij}(\underline{u})) + \lambda \varepsilon_{kk}(\underline{u}))_{,j}$$

$$= (\mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k})_{,j}$$

$$= \mu (u_{i,jj} + u_{j,ji}) + \lambda u_{j,ji}$$

$$= \mu u_{i,jj} + (\mu + \lambda) u_{j,ji}$$

For the time-independent problem we get the Navier/Lamé equations

$$\mu u_{i,jj} + (\mu+\lambda) u_{j,ji} + f_i = 0 .$$

Note that $u_{i,jj} = \Delta u_i$

and $u_{j,j} = \operatorname{div} \underline{u}$.

Hence, the equations are

$$\mu \Delta \underline{u} + (\mu+\lambda) \nabla (\operatorname{div} \underline{u}) + \underline{f} = \underline{0} .$$

It is preferable not to use these forms as they do not "contain the mechanical structure" of the problem. $\square$.

To the equations of motion we still have to add:

The initial conditions at the time $t = 0$:

$$\underline{u}(\underline{x}, 0) = \underline{u}^0(\underline{x}) \quad x \in \Omega,$$

$$\dot{\underline{u}}(\underline{x}, 0) = \underline{v}^0(\underline{x}) \quad x \in \Omega,$$

and

the boundary conditions, e.g. for $\partial\Omega = \Gamma_D \cup \Gamma_N$ (Dirichlet, Neumann):

$$\underline{u}(\underline{x},t) = \underline{g}(\underline{x},t) \qquad \underline{x} \in \Gamma_D.$$

$$\underline{\underline{\sigma}}\,\underline{n} = \underline{t}(\underline{x},t) \qquad x \in \Gamma_N,$$

The 2^{nd} condition is hence

$$(2\mu\,\varepsilon_{ij}(\underline{u}) + \lambda\,\varepsilon_{kk}(\underline{u})\,\delta_{ij})\,n_j = t_i.$$

We arrived at the problem:

Find $\underline{u}(\underline{x},t) = u_i(\underline{x},t)$ such that

$$\rho\,\ddot{u}_i = (2\mu\,\varepsilon_{ij}(\underline{u}) + \lambda\,\varepsilon_{kk}(\underline{u}))_{,j} + f_i$$

in Ω,

$$\left.\begin{aligned} u_i(\underline{x},0) &= u_i^0(\underline{x}) \\ \dot{u}_i(\underline{x},0) &= v_i^0(\underline{x}) \end{aligned}\right\} \quad x \in \Omega.$$

$$u_i(\underline{x},t) = g_i(\underline{x},t), \quad x \in \Gamma_D,$$

$$2\mu\,\varepsilon_{ij}(\underline{u})\,n_j + \lambda\,\varepsilon_{kk}(\underline{u})\,n_j = t_i \quad x \in \Gamma_N.$$

Let us define the elastic energy per unit volume:

$$U_0 = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}(\underline{u})\, \varepsilon_{kl}(\underline{u})$$

and the <u>elastic energy</u> of the body

$$U(\underline{u}) = \frac{1}{2} \int_\Omega C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}\, dx$$

$$= \int_\Omega U_0 \, dx ,$$

The total <u>kinetic energy</u> is

$$K(\underline{u}) = \frac{1}{2} \int_\Omega \rho\, \dot{u}_i \dot{u}_i \, dx$$

$$= \frac{1}{2} \int_\Omega \rho\, |\dot{\underline{u}}|^2 \, dx .$$

The potential energy of the load is

$$\int_\Omega f_i u_i \, dx \quad \left(= \int_\Omega \underline{f} \cdot \underline{u} \, dx \right).$$

We have the following

Theorem. Suppose that the loads f_i and t_i, and the displacement g_i vanish. Then the total energy is constant in time

$$U(\underline{u})(t) + K(\underline{u})(t) = C \neq C(t).$$

Proof. Let us compute:

$$\frac{d}{dt}[U+K] = ?$$

$$\frac{d}{dt} K(\underline{u})(t) = \frac{d}{dt} \frac{1}{2} \int_\Omega \rho \dot{u}_i \dot{u}_i \, dx$$

$$= \frac{1}{2} \int_\Omega \rho (\ddot{u}_i \dot{u}_i + \dot{u}_i \ddot{u}_i) \, dx$$

$$= \int_\Omega \rho \ddot{u}_i \dot{u}_i \, dx .$$

$$\frac{d}{dt} U(\underline{u}) = \frac{1}{2} \frac{d}{dt} \int_\Omega c_{ijkl} \varepsilon_{ij}(\underline{u}) \varepsilon_{kl}(\underline{u}) \, dx$$

$$= \frac{1}{2} \int_\Omega [c_{ijkl} \varepsilon_{ij}(\underline{\dot{u}}) \varepsilon_{kl}(\underline{u}) +$$

$C_{ijkl}\ \varepsilon_{ij}(\underline{u})\ \varepsilon_{kl}(\dot{u})\ ]\ dx$

$$= \int_\Omega C_{ijkl}\ \varepsilon_{kl}(\underline{u})\ \varepsilon_{ij}(\dot{\underline{u}})\ dx$$

(since, we have $C_{ijkl} = C_{klij}$!)

$$= \int_\Omega \sigma_{ij}\ \varepsilon_{ij}(\dot{\underline{u}})\ dx$$

$$= \frac{1}{2} \int_\Omega \sigma_{ij}\ (\dot{u}_{i,j} + \dot{u}_{j,i})\ dx$$

(use the excercise)

$$= \int_\Omega \sigma_{ij}\ \dot{u}_{i,j}\ dx.$$

Next, we "integrate by parts".

$$\int_\Omega \sigma_{ij}\ \dot{u}_{i,j}\ dx$$

$$= -\int_\Omega \sigma_{ij,j}\ \dot{u}_i\ dx + \int_{\partial\Omega} \sigma_{ij} n_j\ \dot{u}_i\ dS$$

$$-\int_\Omega \sigma_{ij,j}\,\dot{u}_i\,dx + \int_{\Gamma_N} \underbrace{\sigma_{ij}n_j}_{=t_i\,=\,0}\,\dot{u}_i\,dS$$

$$+\int_{\Gamma_D} \sigma_{ij}n_j\,\underset{\substack{\| \\ \dot{g}_i\,=\,0}}{\dot{u}_i}\,dS$$

$$= -\int_\Omega \sigma_{ij,j}\,\dot{u}_i\,dS\,.$$

We thus have

$$\frac{d}{dt}\left[K(\underline{u}) + U(\underline{u})\right]$$

$$= \int_\Omega \left[\rho\ddot{u}_i - \sigma_{ij,j}\right]\dot{u}_i\,dx$$

$$= \int_\Omega f_i\,\dot{u}_i\,dx = 0$$

$\square$.

In general we assume that the elastic energy is non-negative

$$\int_\Omega C_{ijkl}\,\varepsilon_{ij}(\underline{u})\,\varepsilon_{kl}(\underline{u})\,dx \geq 0\,.$$

Or stronger, the <u>ellipticity condition</u> $\exists\, \alpha > 0$ s.t.

$$C_{ijkl}\, \varsigma_{ij}\, \varsigma_{kl} \geq \alpha\, \varsigma_{ij}\, \varsigma_{ij}$$
$$= \alpha\, |\underline{\varsigma}|^2 \quad \forall \varsigma_{ij}$$

Since $\mu > 0$, $\lambda \geq 0$ this is true for isotropic materials.

<u>Corollary.</u> The solution to the dynamic problem is unique.

<u>Proof.</u> By linearity we have to show that $\underline{u}^0 = \underline{0}$, $\underline{v}^0 = \underline{0}$, $\underline{f} = \underline{0}$, $\underline{g} = \underline{0}$ and $\underline{t} = \underline{0}$ implies that

$$\underline{u} = \underline{0}.$$

We have

$$U(\underline{u})(0) + K(\underline{u})(0) = 0.$$

and hence

$$U(\underline{u})(t) + K(\underline{u})(t) = 0.$$

That is

$$\frac{1}{2}\int_\Omega c_{ijkl}\,\varepsilon_{ij}(\underline{u})\,\varepsilon_{kl}(\underline{u})\,dx$$

$$+\ \frac{1}{2}\int_\Omega \rho\,\dot{u}_i\,\dot{u}_i\,dx = 0$$

Both integrals are non-negative.

(Due to the ellipticity it holds

$$\int_\Omega c_{ijkl}\,\varepsilon_{ij}(\underline{u})\,\varepsilon_{kl}(\underline{u})\,dx$$

$$\geq \alpha \int_\Omega \varepsilon_{ij}(\underline{u})\,\varepsilon_{ij}(\underline{u}))\,dx \geq 0 \,.)$$

Hence $\underline{u}$ is a rigid body displacement

$$\underline{u}(\underline{x},t) = \underline{a}(t) + \underline{b}(t) \times \underline{x} \,,$$

but independent of time

$$\dot{\underline{u}} = \underline{0} \,.$$

Hence

$$\underline{u}(\underline{x},t) = \underline{a} + b \times \underline{x} \,.$$

The initial condition then implies $\underline{u} \equiv \underline{0}$.