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Integral Equations and Operator Theory

State-feedback stabilization of well-posed linear systems

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Abstract. A finite-dimensional linear time-invariant system is output-stabilizable if and only if it satisfies the *finite cost condition*, i.e., if for each initial state there exists at least one L^2 input that produces an L^2 output. It is exponentially stabilizable if and only if for each initial state there exists at least one L^2 input that produces an L^2 state trajectory. We extend these results to well-posed linear systems with infinite-dimensional input, state and output spaces. Our main contribution is the fact that the stabilizing state feedback is *well posed*, i.e., the map from an exogenous input (or disturbance) to the feedback, state and output signals is continuous in $\mathrm{L}^2_{\mathrm{loc}}$ in both open-loop and closed-loop settings. The state feedback can be chosen in such a way that it also stabilizes the I/O map and induces a (quasi) right coprime factorization of the original transfer function. The solution of the LQR problem has these properties.

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1. Introduction

To illustrate the philosophy behind our results while avoiding undue technicalities, in this introductory section we start with the (more or less well known) finite-dimensional case.

The standard model of a finite-dimensional linear time-invariant system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \qquad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

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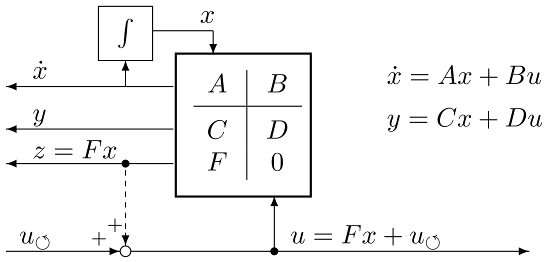

FIGURE 1. State-feedback connection

Here the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ are matrices, and $\mathtt{U} = \mathbb{C}^p$, $\mathtt{X} = \mathbb{C}^n$ and $\mathtt{Y} = \mathbb{C}^q$ are called the *input space*, the *state space* and the *output space*, respectively. We call u the *input* (or *control*), x the *state* and y the *output* of the system.

The *Laplace transform* of u is defined by $\widehat{u}(s) := \int_0^\infty e^{-st} u(t)\, dt$. One easily observes that with zero initial state $x_0 = 0$ equation (1.1) leads to $\widehat{y} = \hat{\mathscr{D}}\widehat{u}$, where

$$\hat{\mathscr{D}}(s) := D + C(sI - A)^{-1}B \tag{1.2}$$

is called the *transfer function* of the system. Conversely, every rational matrix-valued function has a finite-dimensional *realization*, i.e., it is the transfer function of a finite-dimensional system.

State feedback means that we add a second output, say $z(t) = Fx(t)$, where $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$, and feed this signal to the input, as in Figure 1. Under an exogenous input (perturbation) $u_{\circlearrowleft}$, we get $u = Fx + u_{\circlearrowleft}$. Solving for $\dot{x}$, y and z in terms of x and $u_{\circlearrowleft}$, we get the following *closed-loop system* (for $t \geq 0$)

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) + Bu_{\circlearrowleft}(t), \\ y(t) &= (C + DF)x(t) + Du_{\circlearrowleft}(t), \\ z(t) &= Fx(t), \\ x(0) &= x_0. \end{aligned} \tag{1.3}$$

(By the *open-loop system* we mean the original system (1.1) with the additional output $z = Fx$, i.e., as in Figure 1 without the dashed connection.)

The (original) system is called *exponentially stable* iff there exist $M, \epsilon > 0$ such that $\|x(t)\|_{\mathtt{X}} \leq M e^{-\epsilon t} \|x_0\|_{\mathtt{X}}$ $(t \geq 0)$ for each initial state $x_0 \in \mathtt{X}$, or equivalently, iff the spectrum $\sigma(A)$ is contained in the open left half-plane $\mathbb{C}^- := \{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$. By Datko's Theorem, an equivalent condition is that $x \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{X})$ if $u = 0$ (for all $x_0 \in \mathtt{X}$). The system is called *exponentially stabilizable* iff there exists a state feedback operator $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ such that the closed-loop system (1.3) is exponentially stable.

The following condition is called the *state-FCC* (state Finite Cost Condition):

$$\text{For each } x_0 \in \mathtt{X}, \text{ there exists } u \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{U}) \text{ such that } x \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{X}). \tag{1.4}$$

Recall that x stands for the solution of (1.1).

Next we state three theorems for the system (1.1). We shall generalize them in Section 5.

Theorem 1.1. *The state-FCC* (1.4) *holds iff the system is exponentially stabilizable.*

The above result is a special case of the following result that involves also C and D (the terminology will be explained below):

Theorem 1.2. *The output-FCC* (1.5) *holds iff the system is output-stabilizable.*

The *output-FCC* means the following:

$$\text{For each } x_0 \in \mathtt{X}, \text{ there exists } u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U}) \text{ such that } y \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{Y}), \tag{1.5}$$

that is, some *stable* (i.e., L^2) input makes the output stable. This condition is strictly weaker than the state-FCC (1.4).

The system is called *output-stabilizable* if there exists $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$ such that the state-feedback $u(t) = Fx(t)$ makes u and y stable for each initial state $x_0 \in \mathtt{X}$ (with no exogenous input: $u_\circlearrowleft = 0$). In fact, then F can actually be chosen so that u and y become stable for each $x_0 \in \mathtt{X}$ and each $u_\circlearrowleft \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$, and, in addition, the maps $u_\circlearrowleft \mapsto \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ become (right) *coprime*.

Indeed, from (1.3) we obtain, for $x_0 = 0$, that $s\widehat{x}(s) = (A+BF)\widehat{x}(s)+B\widehat{u_\circlearrowleft}(s)$, hence

$$\widehat{x}(s) = (s - A - BF)^{-1}B\widehat{u_\circlearrowleft}(s), \qquad \widehat{y} = \hat{\mathscr{N}}\widehat{u_\circlearrowleft}, \qquad \widehat{u} = \hat{\mathscr{M}}\widehat{u_\circlearrowleft}, \tag{1.6}$$

where $\hat{\mathscr{N}}\widehat{u_\circlearrowleft} = (C + DF)\widehat{x} + D\widehat{u_\circlearrowleft}$ and $\hat{\mathscr{M}}\widehat{u_\circlearrowleft} = \widehat{z} + \widehat{u_\circlearrowleft} = F\widehat{x} + \widehat{u_\circlearrowleft}$, hence

$$\hat{\mathscr{N}}(s) := D+(C+DF)(s-A-BF)^{-1}B, \qquad \hat{\mathscr{M}}(s) := I+F(s-A-BF)^{-1}B. \tag{1.7}$$

By $\left[\begin{smallmatrix} \hat{\mathscr{N}} \\ \hat{\mathscr{M}} \end{smallmatrix}\right] : \widehat{u_\circlearrowleft} \mapsto \left[\begin{smallmatrix} \widehat{y} \\ \widehat{u} \end{smallmatrix}\right]$ being *right coprime* we mean that there exist $f, g \in \mathrm{H}^\infty$ that satisfy the *Bézout equation* $f\hat{\mathscr{M}} + g\hat{\mathscr{N}} \equiv I$. Recall that H^∞ denotes the space of bounded holomorphic functions on the right half-plane $\mathbb{C}^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$.

We express the above as follows:

Theorem 1.3. *If the output-FCC* (1.5) *holds, then there exists an output- and I/O-stabilizing state feedback with* $\hat{\mathscr{N}}$ *and* $\hat{\mathscr{M}}$ *right coprime.*

By *output- and I/O-stabilizing* we mean that (see Figure 1)

$$\| \left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right] \|_2 \le M(\|x_0\|_{\mathtt{X}} + \|u_\circlearrowleft\|_2) \quad (x_0 \in \mathtt{X},\ u_\circlearrowleft \in \mathrm{L}^2). \tag{1.8}$$

The map $\hat{\mathscr{D}} : \widehat{u} \mapsto \widehat{y}$ can be written as $\hat{\mathscr{D}} := \hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$. We express this as follows:

Corollary 1.4. *Any function having an output-stabilizable realization has a right-coprime factorization.*

The state-feedback operator F used above is usually obtained by solving the so-called (algebraic) LQR Riccati equation

$$\mathcal{P}B(I + DD^*)^{-1}B^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C, \tag{1.9}$$

since $F = -(I + DD^*)^{-1}B^*\mathcal{P}$ for the minimal nonnegative solution $\mathcal{P}$. This F is the unique state-feedback operator that minimizes the *LQR cost function*

$$\mathcal{J}(x_0, u) = \int_0^\infty \left(\|y(t)\|_{\mathtt{Y}}^2 + \|u(t)\|_{\mathtt{U}}^2\right) dt \tag{1.10}$$

for each initial state $x_0 \in \mathtt{X}$.

In Section 5 we shall extend the above theorems to arbitrary *well-posed linear systems (WPLSs)*. These systems are a generalization of the systems of type (1.1) and allow unbounded generators and infinite-dimensional input, state and output spaces; see Section 2 for details.

The state-FCC has been studied under the name "optimizability" in, e.g., [WR01]. Thus, our generalization of Theorem 1.1 shows that optimizability is equivalent to exponential stabilizability.

It was already known that under the output-FCC an output-stabilizing control is produced by another WPLS, as shown in [Zwa96] (the case with $C \in \mathcal{B}(\mathtt{X}, \mathtt{Y})$ was shown in [FLT88]), or by "ill-posed state feedback". It was not known that this other system can be obtained from the original one by a *well-posed state feedback*. It means that the state-feedback loop is well posed with respect to external disturbance, i.e., that the maps $\hat{\mathscr{M}} : \widehat{u_\circlearrowleft} \mapsto \widehat{u}$ and $\hat{\mathscr{M}}^{-1}$ are well defined (see Figure 1). In fact, the functions $\hat{\mathscr{M}}$ and $\hat{\mathscr{M}}^{-1}$ become proper (bounded on some right half-plane).

Any *bounded* state-feedback operator ($F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$) generates a well-posed state feedback, but so do some unbounded ones. If B and C are bounded, then the stabilizing state feedback is given by a bounded F; this special case of Theorems 1.1 and 1.2 was already known. The two theorems were known also for fairly unbounded B's in the case that A generates an analytic semigroup and C is bounded [LT00].

To extend Theorem 1.3 and Corollary 1.4 to arbitrary WPLSs, we must replace coprimeness by "quasi-coprimeness", which we define below.

For any $\omega \in \mathbb{R}$ we set $\mathbb{C}_\omega^+ := \{s \in \mathbb{C} \mid \operatorname{Re} s > \omega\}$. By $\mathrm{H}_\omega^2(\mathtt{U})$ we denote the Hilbert space of holomorphic functions $\mathbb{C}_\omega^+ \to \mathtt{U}$ for which

$$\|h\|_{\mathrm{H}_\omega^2}^2 := \sup_{r>\omega} \int_{-\infty}^\infty \|h(r+it)\|_{\mathtt{U}}^2 \, dt < \infty. \tag{1.11}$$

Moreover, $\mathbb{C}^+ := \mathbb{C}_0^+$, $\mathrm{H}^2 := \mathrm{H}_0^2$. Bounded holomorphic functions $\hat{\mathscr{N}} : \mathbb{C}^+ \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ and $\hat{\mathscr{M}} : \mathbb{C}^+ \to \mathcal{B}(\mathtt{U})$ are called *quasi–right coprime* iff

$$\begin{bmatrix} \hat{\mathscr{N}} \\ \hat{\mathscr{M}} \end{bmatrix} h \in \mathrm{H}^2 \;\Rightarrow\; h \in \mathrm{H}^2 \quad \text{for every } h \in \mathrm{H}_\omega^2(U) \text{ and } \omega \in \mathbb{R}. \tag{1.12}$$

In other words, quasi–right coprimeness means that if h is in some H_ω^2 and its image is in H^2, then h must actually have been in H^2. We identify any function with its holomorphic extension (if any) to a right half-plane, so "$\in \mathrm{H}^2$" means that "is the restriction of an element of H^2".

For a quasi–right coprime factorization $\hat{\mathscr{D}} = \hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$, the image $\left[\begin{smallmatrix}\mathscr{M}\\ \mathscr{N}\end{smallmatrix}\right][\mathrm{H}^2]$ equals the *graph* $\left[\begin{smallmatrix}I\\ \hat{\mathscr{D}}\end{smallmatrix}\right][\{f \in \mathrm{H}^2 \,|\, \hat{\mathscr{D}}f \in \mathrm{H}^2\}]$. In fact, also the converse holds. See also Lemma 4.4.

Whenever $\hat{\mathscr{N}}$ and $\hat{\mathscr{M}}$ are (Bézout) right coprime, they are quasi–right coprime. Indeed, $\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right] h \in \mathrm{H}^2$ and $f\hat{\mathscr{M}} + g\hat{\mathscr{N}} \equiv I$ imply that $h = \begin{bmatrix} f & g\end{bmatrix}\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right] h \in \mathrm{H}^2$. The two forms of coprimeness are equivalent if $\hat{\mathscr{N}}$ and $\hat{\mathscr{M}}$ are rational [Mik02, Lemma 6.5.3]. However, quasi coprimeness is in certain sense a more natural extension of coprimeness to the non-rational case. We shall treat different forms of coprimeness in detail in future articles.

Our proof is based on showing that the control minimizing (1.10) is given by well-posed state feedback, but due to the unboundedness of B, C and F, we must use the integral Riccati equations of Lemma 3.8 instead of the algebraic one above. Those equations allow us to reduce the minimization problem to a stable one by replacing A by $A-\alpha$ for α big enough (we must add some cost on the state to keep the minimal cost the same). The stable LQR problem can then be solved by using a spectral factorization.

See Section 6 for generalizations and further historical comments.

2. Well-posed linear systems and state feedback

In this section we present our notation and definitions except for those concerning optimization and coprimeness. The definitions and claims in this section and further details can be found in, e.g., [Sta05], [Sta98a], [Wei94] or [Mik02, Chapter 6].

By $\mathcal{B}(\mathtt{U},\mathtt{Y})$ we denote the set of bounded linear operators $\mathtt{U} \to \mathtt{Y}$, and we write $\mathcal{B}(\mathtt{U}) := \mathcal{B}(\mathtt{U},\mathtt{U})$ (similarly for something else in place of $\mathcal{B}$).

Let $\mathtt{U},\mathtt{X},\mathtt{Y}$ be arbitrary complex Hilbert spaces. If the generators of the system (1.1) are *bounded*, i.e., $\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right] \in \mathcal{B}(\mathtt{X}\times\mathtt{U},\mathtt{X}\times\mathtt{Y})$, then the unique solution of (1.1) is obviously given by the system

$$\begin{cases} x(t) &= \mathscr{A}^t x_0 + \mathscr{B}^t u \\ y &= \mathscr{C} x_0 + \mathscr{D} u, \end{cases} \tag{2.1}$$

where

$$\begin{aligned} \mathscr{A}^t &= \mathrm{e}^{At}, \qquad & \mathscr{B}^t u &= \int_0^t \mathscr{A}^{t-s} B u(s)\, ds, \\ (\mathscr{C}x_0)(t) &= C\mathscr{A}^t x_0, \quad & (\mathscr{D}u)(t) &= C\mathscr{B}^t u + Du(t). \end{aligned} \tag{2.2}$$

The above formulas are actually valid for fairly unbounded operators, but in the most general case the right-hand-sides (at least "$C\mathscr{B}^t$") become meaningless. Therefore, the WPLSs (also known as Salamon–Weiss systems or abstract linear systems) were defined by requiring the system to be linear and time-invariant and $\mathscr{A}$ to be strongly continuous; in addition, one requires that $\left[\begin{smallmatrix}\mathscr{A}^t & \mathscr{B}^t\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ is causal and continuous $\mathtt{X}\times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathtt{U}) \to \mathtt{X}\times \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathtt{Y})$ for each $t \ge 0$, or equivalently, that

$$\|x(t)\|_{\mathtt{X}}^2 + \int_0^t \|y(s)\|_{\mathtt{Y}}^2\, ds \le K_t\big(\|x_0\|_{\mathtt{X}}^2 + \int_0^t \|u(s)\|_{\mathtt{U}}^2\, ds\big) \tag{2.3}$$

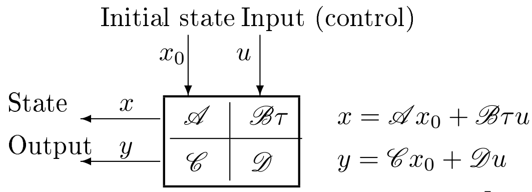

FIGURE 2. Input/state/output diagram of a WPLS $\left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$

for all (equivalently, some) $t > 0$, where K_t depends on t only. An equivalent formulation, from [Sta98a], is given in Definition 2.1. There we use the natural extensions $\mathscr{B}$ (of $\mathscr{B}^t\tau^{-t}$) and $\mathscr{D}$ that allow the inputs to be defined on the whole real line, thus simplifying some formulae.

We use the notation $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2 = \{f \mid \mathrm{e}^{-\omega\cdot}f \in \mathrm{L}^2\}$, $(\tau^t u)(s) := u(t+s)$ and $\pi_\pm u := \chi_{\mathbb{R}_\pm} u$, where $\chi_E(t) := \begin{cases}1, & t \in E;\\ 0, & t \notin E\end{cases}$. When $E \subset \mathbb{R}$, we set $\pi_E u := \chi_E u$. We identify $\mathrm{L}^2_\omega(E;\mathtt{U})$ with functions in $\mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ that vanish outside E.

We study the following generalization of systems of type (2.2):

Definition 2.1 (WPLS and stability). Let $\omega \in \mathbb{R}$. An *ω-stable well-posed linear system on* $(\mathtt{U},\mathtt{X},\mathtt{Y})$ is a quadruple $\Sigma = \left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$, where $\mathscr{A}^t$, $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ are bounded linear operators of the following type:

1. $\mathscr{A}^\cdot$ is a strongly continuous semigroup of bounded linear operators on $\mathtt{X}$ satisfying $\sup_{t\ge0}\|\mathrm{e}^{-\omega t}\mathscr{A}^t\|_{\mathtt{X}} < \infty$;
2. $\mathscr{B}\colon \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathtt{X}$ satisfies $\mathscr{A}^t\mathscr{B}u = \mathscr{B}\tau^t\pi_- u$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ and $t \in \mathbb{R}_+$;
3. $\mathscr{C}\colon \mathtt{X} \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ satisfies $\mathscr{C}\mathscr{A}^t x = \pi_+\tau^t\mathscr{C}x$ for all $x \in \mathtt{X}$ and $t \in \mathbb{R}_+$;
4. $\mathscr{D}\colon \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U}) \to \mathrm{L}^2_\omega(\mathbb{R};\mathtt{Y})$ satisfies $\tau^t\mathscr{D}u = \mathscr{D}\tau^t u$, $\pi_+\mathscr{D}\pi_- u = \mathscr{C}\mathscr{B}u$, and $\pi_-\mathscr{D}\pi_+ u = 0$ for all $u \in \mathrm{L}^2_\omega(\mathbb{R};\mathtt{U})$ and $t \in \mathbb{R}$.

The different components of $\Sigma = \left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$ are named as follows: $\mathscr{A}$ is the *semigroup*, $\mathscr{B}$ the *input map*, $\mathscr{C}$ the *output map*, and $\mathscr{D}$ the *I/O map* (input/output map) of Σ.

We say that $\mathscr{A}$ (resp. $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$) is *stable* if 1. (resp. 2., 3., 4.) holds for $\omega = 0$. *Exponentially stable* means ω-stable for some $\omega < 0$. The system is *output stable* (resp. *I/O-stable*) if $\mathscr{C}$ (resp. $\mathscr{D}$) is stable.

We set $\Sigma^\tau := \left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\tau\\ \hline \mathscr{C} & \mathscr{D}\end{array}\right]$, $\mathscr{B}^t := \mathscr{B}\tau^t\pi_+ = \mathscr{B}\tau^t\pi_{[0,t)}$, $\mathscr{D}^t := \pi_{[0,t)}\mathscr{D}\pi_{[0,t)}$.

For any $x_0 \in \mathtt{X}$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathtt{U})$ we associate the *state* (trajectory) $x := \mathscr{A}x_0 + \mathscr{B}\tau u$ and *output* $y := \mathscr{C}x_0 + \mathscr{D}u$ on $\mathbb{R}_+$ (i.e., $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] = \Sigma^\tau\left[\begin{smallmatrix}x_0\\ u\end{smallmatrix}\right]$), as in (2.1) and Figure 2. (By causality, also $\mathscr{D}$ is defined for any $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathtt{U})$ through $\pi_{[0,t)}\mathscr{D}u = \pi_{[0,t)}\mathscr{D}\pi_{[0,t)}u$ $(t \ge 0)$.)

From Definition 2.1 we easily obtain that $\mathscr{B} = \mathscr{B}\pi_-$, $\mathscr{C} = \pi_+\mathscr{C}$ and

$$\pi_+\tau^t y = \mathscr{C}x(t) + \mathscr{D}\pi_+\tau^t u \quad (t \ge 0). \tag{2.4}$$

Indeed, $\pi_+\tau^t y = \pi_+\tau^t\mathscr{C}x_0 + \pi_+\mathscr{D}(\pi_- + \pi_+)\tau^t u = \mathscr{C}\mathscr{A}^t x_0 + \mathscr{C}\mathscr{B}\tau^t u + \mathscr{D}\pi_+\tau^t u = \mathscr{C}x(t) + \mathscr{D}\pi_+\tau^t u$. This says that the output is "time-invariant", i.e., "the remaining output (at time t) depends only on the current state and the remaining input".

The same holds for the state: $\pi_+\tau^t x = \mathscr{A}x(t)+\mathscr{B}\tau\pi_+\tau^t u$ ($t \ge 0$). Conversely, any linear system satisfying these two equations and (2.3) is a (restriction of a) WPLS.

By A we denote the infinitesimal generator of $\mathscr{A}$. One can show that there exist $B \in \mathcal{B}(\mathtt{U}, \mathrm{Dom}(A^*)^*)$ and $C \in \mathcal{B}(\mathrm{Dom}(A), \mathtt{Y})$ such that the middle formulas in (2.2) hold (for $u \in \mathrm{L}^2_\omega$ and $x_0 \in \mathrm{Dom}(A)$). Moreover, $\dot{x} = Ax + Bu$ in $\mathrm{Dom}(A^*)^*$.[1]

An ω-stable WPLS is an ω'-stable WPLS for any $\omega' > \omega$ (we identify the unique extensions/restrictions of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ for different ω).

Exponential stability of a system is equivalent to that of its semigroup, hence Datko's Theorem [Dat70] leads to the following:

Lemma 2.2. *The WPLS* $\Sigma = \left[\begin{smallmatrix}\mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ *is exponentially stable iff* $\mathscr{A}x_0 \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{X})$ *for all* $x_0 \in \mathtt{X}$. □

Exponential stability implies that $\sigma(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. The converse holds if, e.g., A is bounded or analytic [Sta05].

Now it is the time to present our assumptions:

Standing Assumption 2.3. Throughout this article we assume that $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ are complex Hilbert spaces, $\Sigma = \left[\begin{smallmatrix}\mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D}\end{smallmatrix}\right]$ is a WPLS on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$, and $J = J^* \in \mathcal{B}(\mathtt{Y})$.

(The operator J will not be needed before Section 3.)

Let $\omega \in \mathbb{R}$. We define $\mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y})$ to be the (closed) subspace of operators $\mathscr{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbb{R}; \mathtt{U}); \mathrm{L}^2_\omega(\mathbb{R}; \mathtt{Y}))$ that are *causal* (i.e., $\pi_-\mathscr{D}\pi_+ = 0$) and *time-invariant* (i.e. $\tau^t\mathscr{D} = \mathscr{D}\tau^t$ for all $t \in \mathbb{R}$). The I/O maps of WPLSs are exactly all such operators ($\mathrm{TIC}_\infty(\mathtt{U}, \mathtt{Y}) := \cup_{\omega\in\mathbb{R}}\mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y})$). The Laplace transform $u \mapsto \widehat{u}$ is an isometric (modulo $\sqrt{2\pi}$) isomorphism of L^2_ω onto H^2_ω. By $\mathrm{H}^\infty(\Omega; \mathtt{X})$ we denote the Banach space of bounded holomorphic functions $\Omega \to \mathtt{X}$ with supremum norm. We set $\mathbb{C}^+_\omega := \{s \in \mathbb{C} \mid \operatorname{Re} s > \omega\}$.

For each $\mathscr{D} \in \mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y})$, there exists a unique function $\widehat{\mathscr{D}} \in \mathrm{H}^\infty_\omega(\mathtt{U}, \mathtt{Y}) := \mathrm{H}^\infty(\mathbb{C}^+_\omega; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$, called the *transfer function* of $\mathscr{D}$, such that $\widehat{\mathscr{D}u} = \widehat{\mathscr{D}}\hat{u}$ on $\mathbb{C}^+_\omega$ for every $u \in \mathrm{L}^2_\omega(\mathbb{R}_+; \mathtt{U})$. The mapping $\mathscr{D} \mapsto \hat{\mathscr{D}}$ is an isometric isomorphism of $\mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y})$ onto $\mathrm{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$. If B is bounded, then $\hat{\mathscr{D}}(s) = D + C(s - A)^{-1}B$.

A function is called *proper* if it is bounded and holomorphic on some right half-plane. Thus, $\mathrm{H}^\infty_\infty(\mathtt{U}, \mathtt{Y}) := \cup_{\omega\in\mathbb{R}}\mathrm{H}^\infty_\omega(\mathtt{U}, \mathtt{Y})$ is the set of all proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued functions. (We identify functions that coincide on some right half-plane.)

By $\mathcal{G}$ we denote the group of invertible elements. Thus, e.g., $\mathcal{G}\mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y})$ stands for $\{\mathscr{D} \in \mathrm{TIC}_\omega(\mathtt{U}, \mathtt{Y}) \mid \mathscr{D}^{-1} \in \mathrm{TIC}_\omega(\mathtt{Y}, \mathtt{U})\}$, i.e., it corresponds to $\mathcal{G}\mathrm{H}^\infty_\omega$, the set of bounded holomorphic functions $\hat{\mathscr{D}} : \mathbb{C}^+_\omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ for which $\hat{\mathscr{D}}^{-1}$ exists and is (uniformly) bounded.

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[1] This is based on the fact that $A : \mathrm{Dom}(A) \to \mathtt{X}$ extends to a continuous map $\mathtt{X} \to \mathrm{Dom}(A^*)^*$ and generates a semigroup on $\mathrm{Dom}(A^*)^*$. Similarly, $A_{|\,\mathrm{Dom}(A^2)}$ generates a semigroup on $\mathrm{Dom}(A)$, and all three semigroups are isomorphic. It also follows that $(s - A)^{-1}B$ becomes well defined for $s \in \rho(A_{|\mathtt{X}}) = \rho(A_{|\,\mathrm{Dom}(A)})$. However, we only need these A's for examples.

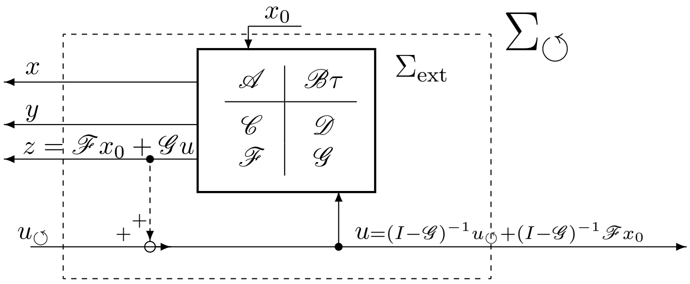

FIGURE 3. State-feedback connection for a general WPLS

Now we shall return to state feedback. When the input operator B is bounded, the generators $\left(\begin{smallmatrix} A & B \\ \hline C & D \\ F & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A+BF & B \\ \hline C+DF & D \\ F & 0 \end{smallmatrix}\right)$, determine the open-loop and closed-loop systems, respectively, corresponding to a state feedback operator F, as in Figure 1 and equations (1.3).

Thus, state feedback means adding an extra output to the system and feeding that output back to the input (like $z(t) = Fx(t)$ in Figure 1).

For a general WPLS, the definition is the same: a state feedback means a pair $[\ \mathscr{F} \mid \mathscr{G}\ ]$ such that the extended system Σ_{ext} in Figure 3 is a WPLS.

The state feedback is called *admissible* if $(I - \hat{\mathscr{G}})^{-1}$ exists and is proper, or equivalently, if the map $(I - \mathscr{G}) : u \mapsto u_{\circlearrowleft}$ has a bounded and causal inverse on L^2_ω for some $\omega \in \mathbb{R}$. This means that the (closed) state-feedback loop is well posed under external disturbance.

Thus, the definition of an admissible (or "well-posed" or "proper") state feedback is as follows:

Definition 2.4 ($\Sigma_{\circlearrowleft}, [\ \mathscr{F} \mid \mathscr{G}\ ]$). A pair $[\ \mathscr{F} \mid \mathscr{G}\ ]$ is called an *admissible state-feedback pair* for Σ if the extended system

$$\Sigma_{\text{ext}} := \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B} \\ \hline \mathscr{C} & \mathscr{D} \\ \mathscr{F} & \mathscr{G} \end{array}\right] \tag{2.5}$$

is a WPLS and $I - \mathscr{G} \in \mathcal{G}\mathrm{TIC}_\infty(\mathtt{U})$.

We set $\mathscr{M} := (I - \mathscr{G})^{-1}$, $\mathscr{N} := \mathscr{D}\mathscr{M}$, and denote the corresponding closed-loop system (see Figure 3) by

$$\Sigma^\tau_{\circlearrowleft} = \left[\begin{array}{c|c} \mathscr{A}_{\circlearrowleft} & \mathscr{B}_{\circlearrowleft}\tau \\ \hline \mathscr{C}_{\circlearrowleft} & \mathscr{D}_{\circlearrowleft} \\ \mathscr{F}_{\circlearrowleft} & \mathscr{G}_{\circlearrowleft} \end{array}\right] = \left[\begin{array}{c|c} \mathscr{A} + \mathscr{B}\tau\mathscr{M}\mathscr{F} & \mathscr{B}\mathscr{M}\tau \\ \hline \mathscr{C} + \mathscr{D}\mathscr{M}\mathscr{F} & \mathscr{D}\mathscr{M} \\ \mathscr{M}\mathscr{F} & \mathscr{M} - I \end{array}\right] \tag{2.6}$$

$$= \Sigma^\tau_{\text{ext}} \begin{bmatrix} I & 0 \\ -\mathscr{F} & I - \mathscr{G} \end{bmatrix}^{-1} = \Sigma^\tau_{\text{ext}} \begin{bmatrix} I & 0 \\ \mathscr{M}\mathscr{F} & \mathscr{M} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_{\circlearrowleft} \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ u - u_{\circlearrowleft} \end{bmatrix}. \tag{2.7}$$

We call $[\,\mathscr{F} \,|\, \mathscr{G}\,]$ *exponentially stabilizing* if $\Sigma_\circlearrowleft$ is exponentially stable. If there exists an exponentially stabilizing state-feedback pair for Σ, then Σ is called *exponentially stabilizable* (similarly for output-stabilizing or I/O-stabilizing).

(The system $\Sigma_\circlearrowleft$ is necessarily a WPLS. Note that $\Sigma_\circlearrowleft$ is output stable iff $\mathscr{C}_\circlearrowleft$ and $\mathscr{F}_\circlearrowleft$ map $\mathtt{X}$ into L^2.)

Any $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$ determines an admissible state feedback (with $\mathscr{F} = F\mathscr{A}$, $\mathscr{G} = F\mathscr{B}\tau$), but so do also some unbounded operators. The above definition also allows for feedthrough terms (i.e., $z(t) = Fx(t) + Gu(t)$ or $\mathscr{G} = F\mathscr{B}\tau + G$, where $G \in \mathcal{B}(\mathtt{U})$), but if $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$, then admissibility is equivalent to $I - G \in \mathcal{GB}(\mathtt{U})$, and essentially the same feedback is obtained by using the state-feedback operator $(I-G)^{-1}F$ with zero feedthrough (see Lemma A.5). However, in the case that F is unbounded, the feedthrough term $G = \lim_{s\to+\infty}\hat{\mathscr{G}}(s)$ need not exist [WW97, Example 11.5].

3. Optimal control and Riccati equations

In this section we shall present certain necessary and sufficient conditions in terms of Riccati equations for a control to be optimal. In Section 4, these conditions will be applied to establish the existence of a (1.10)-minimizing state feedback.

The cost (1.10) is finite iff $u \in \mathcal{U}(x_0)$, where

$$\mathcal{U}(x_0) := \{u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U}) \,|\, y \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{Y})\}. \tag{3.1}$$

We call this the set of *admissible* controls. Note that the output-FCC (1.5) holds iff $\mathcal{U}(x_0) \neq \emptyset$ for each $x_0 \in \mathtt{X}$, hence the name finite cost condition.

The following is obvious (and given in [Zwa96]):

Lemma 3.1. *The set $\mathcal{U}(0)$ is a subspace of L^2. If $u \in \mathcal{U}(x_0)$, then $\mathcal{U}(x_0) = u + \mathcal{U}(0)$.* □

A control u_{opt} is called *$\mathcal{J}$-minimizing* for x_0 if $\mathcal{J}(x_0, u_{\mathrm{opt}}) \le \mathcal{J}(x_0, u)$ for every $u \in \mathcal{U}(x_0)$. It is well known (see [Zwa96, the proof of Theorem 6]) that under the output-FCC a (1.10)-minimizing control exists:

Lemma 3.2. *Assume the output-FCC (1.5). Define $\mathcal{J}$ by (1.10).*

Then there exists a unique $\mathcal{J}$-minimizing control $u_{\mathrm{opt}}^{x_0}$ for every $x_0 \in \mathtt{X}$ and a nonnegative operator $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ such that $\mathcal{J}(x_0, u_{\mathrm{opt}}^{x_0}) = \langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$ for every $x_0 \in \mathtt{X}$. □

In Section 4 we shall show that the above minimizing control is given by admissible state feedback. For that purpose we need certain "integral" Riccati equation conditions for minimality, also over other cost functions than (1.10) (see Lemma 3.9). Therefore, we shall introduce a cost operator $J = J^* \in \mathcal{B}(\mathtt{Y})$ and the (generalized) *cost function*

$$\mathcal{J}(x_0, u) := \langle y, Jy\rangle_{\mathrm{L}^2} = \int_0^\infty \langle y(t), Jy(t)\rangle_{\mathtt{Y}}\,dt \quad (x_0 \in \mathtt{X},\ u \in \mathcal{U}(x_0)). \tag{3.2}$$

As a by-product, our proofs and formulas actually apply in a much more general optimization setting (possibly indefinite, such as the "minimax H^∞ control" of [Sta98c] and [Mik02]). The explicit inclusion of u in (3.2) would not only reduce generality (see Lemma 3.4) but also lengthen numerous formulae below by half.

A control $u \in \mathcal{U}(x_0)$ is called *J-optimal* for x_0 (and Σ) if $\langle y, J\mathscr{D}\eta\rangle_{L^2} = 0$ for each $\eta \in \mathcal{U}(0)$. (By [Mik02, Lemma 8.3.6], this corresponds to a zero of the Fréchet derivative of $\langle y, Jy\rangle_{L^2}$.)

When $J = I$, this orthogonality condition implies that y is of minimal norm. More generally:

Lemma 3.3. *A control u minimizes $\mathcal{J}(x_0, \cdot)$ over $\mathcal{U}(x_0)$ iff u is J-optimal for x_0 and $\mathcal{J}(0, \cdot) \geq 0$.*

Proof. The sufficiency follows from Lemma 3.1 and the fact that for any J-optimal u for x_0 we have

$$\mathcal{J}(x_0, u + \eta) = \mathcal{J}(x_0, u) + \mathcal{J}(0, \eta) \quad (\eta \in \mathcal{U}(0)), \tag{3.3}$$

which follows from the identity $\langle y + \mathscr{D}\eta, J(y + \mathscr{D}\eta)\rangle = \langle y, Jy\rangle + 0 + 0 + \langle \mathscr{D}\eta, J\mathscr{D}\eta\rangle$.

If $\langle y, J\mathscr{D}\eta\rangle \neq 0$ for some $\eta \in \mathcal{U}(0)$, then $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{J}(x_0, u + \alpha\eta)$ is nonzero at $\alpha = 0$, hence J-optimality is also necessary. □

The cost (1.10) is a special case of (3.2):

Lemma 3.4. *Set $\tilde{\mathscr{C}} := \left[\begin{smallmatrix}\mathscr{C}\\ 0\end{smallmatrix}\right]$, $\tilde{\mathscr{D}} := \left[\begin{smallmatrix}\mathscr{D}\\ I\end{smallmatrix}\right]$. Then $\tilde{\Sigma} := \left[\begin{smallmatrix}\mathscr{A} & \mathscr{B}\\ \tilde{\mathscr{C}} & \tilde{\mathscr{D}}\end{smallmatrix}\right]$ is a WPLS on $(\mathtt{U}, \mathtt{X}, \mathtt{Y} \times \mathtt{U})$. Moreover, a control is I-optimal for $\tilde{\Sigma}$ iff it is (1.10)-minimizing.* □

Indeed, $\tilde{\Sigma}$ has the output $\tilde{y} := \tilde{\mathscr{C}}x_0 + \tilde{\mathscr{D}}u = \left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right]$, hence $\mathcal{J}_{\tilde{\Sigma}}(x_0, u) := \langle \tilde{y}, I\tilde{y}\rangle = \|y\|_2^2 + \|u\|_2^2$, so the optimality claim follows from Lemma 3.3 (applied to $\tilde{\Sigma}$ and I; note that $\mathcal{U}(x_0)$ is the same for both Σ and $\tilde{\Sigma}$). The WPLS claim is obvious.

Naturally, a minimal cost is unique (for any fixed $x_0 \in \mathtt{X}$). In fact, the "J-optimal cost" is unique also for indefinite J:

Lemma 3.5. *If u and v are J-optimal controls for $x_0 \in \mathtt{X}$, then $\mathcal{J}(x_0, u) = \mathcal{J}(x_0, v)$.*

Proof. By Lemma 3.1, $\tilde{u} := v - u \in \mathcal{U}(0)$. But $\langle y + \mathscr{D}\tilde{u}, J\mathscr{D}\eta\rangle = 0$ $(\eta \in \mathcal{U}(0))$, hence $\langle \mathscr{D}\tilde{u}, J\mathscr{D}\eta\rangle = 0$, also for $\eta = \tilde{u}$. This and (3.3) imply that $\mathcal{J}(x_0, u + \tilde{u}) = \mathcal{J}(x_0, u)$. □

When using the "dynamic programming principle", we need the following:

Lemma 3.6. *Let $x_0 \in \mathtt{X}$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathtt{U})$. Then $u \in \mathcal{U}(x_0)$ iff $\pi_+\tau^t u \in \mathcal{U}(\mathscr{A}^t x_0 + \mathscr{B}^t u)$ for some (equivalently, all) $t \geq 0$.*

This says that u is admissible for some initial state $x(0) = x_0$ iff at some (hence any) moment t the remaining part of u is admissible for the current state $x(t)$.

Proof. Obviously, $u \in L^2$ iff $\pi_+\tau^t u \in L^2$. By (2.4), $y \in L^2$ iff $\mathscr{C}x(t) + \mathscr{D}\pi_+\tau^t u \in L^2$, hence $u \in \mathcal{U}(x_0)$ iff $\pi_+\tau^t u \in \mathcal{U}(x(t))$. □

The ("Riccati") operator $\mathcal{P}$ in Lemma 3.2 is called the J-optimal cost operator:

Definition 3.7. We call $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ the *J-optimal cost operator* for Σ if, for each $x_0 \in \mathtt{X}$, there exists at least one J-optimal control u with $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$.

Obviously, then the output-FCC holds and $\mathcal{P} = \mathcal{P}^*$. By Lemma 3.5, then $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$ for every x_0 and every J-optimal u, hence $\mathcal{P}$ is unique.

We call $\omega_A := \inf_{t>0} \left(t^{-1} \log \|\mathscr{A}^t\|\right)$ the *growth rate* of $\mathscr{A}$. By [Sal89, Lemma 2.1], the whole system Σ is ω-stable for any $\omega > \omega_A$.

Now we can derive certain necessary and/or sufficient conditions for $\mathcal{P}$ and for J-optimal controls. The conditions (3.5) and (3.8) below are integral versions of the standard algebraic Riccati equation (if, e.g., B or C is bounded, we can differentiate the integral equations to obtain the algebraic ones; see [Mik02, Sections 9.11&9.7]). The other, non-standard "Riccati" equations with parameter $r \in \mathbb{R}$ will be used later below to reduce the optimization of Σ to optimization of another, stable system. The convergence conditions (3.7) and (3.9) can be used to distinguish the "stabilizing solution" of the Riccati equation from other solutions [Mik02].

Lemma 3.8 (Riccati equations). *Assume that the J-optimal cost operator $\mathcal{P}$ exists. Let $x_0, x_1 \in \mathtt{X}$ and $r \in \mathbb{R}$. Let $u \in \mathcal{U}(x_0)$ be arbitrary and recall that $x := \mathscr{A}x_0 + \mathscr{B}\tau u$, $y := \mathscr{C}x_0 + \mathscr{D}u$.*

(a): *If u_k is a J-optimal control for x_k $(k = 0, 1)$, then*

$$\langle \mathscr{C}x_1 + \mathscr{D}u_1, J(\mathscr{C}x_0 + \mathscr{D}u_0)\rangle_{\mathrm{L}^2} = \langle x_1, \mathcal{P}x_0\rangle_{\mathtt{X}}. \tag{3.4}$$

(b): *If u is a J-optimal control for x_0, then $\pi_+\tau^t u$ is J-optimal for $x(t)$ and* (3.5)–(3.11) *hold.*

$$\langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}} = \langle y, \pi_{[0,t)} J y\rangle_{\mathrm{L}^2} + \langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}} \qquad \forall t \geq 0. \tag{3.5}$$

$$\langle y, Jy\rangle_{\mathrm{L}^2} = \langle \left[\begin{smallmatrix} y \\ x\end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & 2r\mathcal{P}\end{smallmatrix}\right] \left[\begin{smallmatrix} y \\ x\end{smallmatrix}\right]\rangle_{\mathrm{L}^2_r} \qquad \text{if } 0 \leq r > \omega_A. \tag{3.6}$$

$$\langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}} \overset{t\to+\infty}{\to} 0. \tag{3.7}$$

(c): *The control $u \in \mathcal{U}(x_0)$ is J-optimal for x_0 iff* (3.8) *and* (3.9) *hold.*

$$\langle x(t), \mathcal{P}\mathscr{B}^t\eta\rangle_{\mathtt{X}} = -\langle y, J\mathscr{D}^t\eta\rangle_{\mathrm{L}^2} \qquad \forall t \geq 0,\ \eta \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathtt{U}), \tag{3.8}$$

$$\langle x(t), \mathcal{P}\mathscr{B}^t\eta\rangle_{\mathtt{X}} \overset{t\to+\infty}{\longrightarrow} 0 \qquad \forall \eta \in \mathcal{U}(0). \tag{3.9}$$

(d): *We have* (3.8)$\Leftrightarrow$(3.10).

$$-\mathrm{e}^{-2rt}\langle x(t), \mathcal{P}\mathscr{B}^t\eta\rangle_{\mathtt{X}} = \langle y, \pi_{[0,t)} J\mathscr{D}\eta\rangle_{\mathrm{L}^2_r} + 2r\langle x, \pi_{[0,t)}\mathcal{P}\mathscr{B}\tau\eta\rangle_{L^2_r} \\ \forall t \geq 0,\ \eta \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathtt{U}). \tag{3.10}$$

(e): *We have* (3.5)$\Leftrightarrow$(3.11).

$$\langle y, \pi_{[0,t)} J y\rangle_{\mathrm{L}^2} + \langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}} \\ = \langle \left[\begin{smallmatrix} y \\ x\end{smallmatrix}\right], \pi_{[0,t)} \left[\begin{smallmatrix} J & 0 \\ 0 & 2r\mathcal{P}\end{smallmatrix}\right] \left[\begin{smallmatrix} y \\ x\end{smallmatrix}\right]\rangle_{\mathrm{L}^2_r} + \mathrm{e}^{-2rt}\langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}} \quad \forall t \geq 0. \tag{3.11}$$

Before the proof, we give here some kind of intuitive explanations for the above "Riccati" equations. Part (a) says that $\langle y_1, Jy_0\rangle = \langle x_1, \mathcal{P}x_0\rangle$. If $J = I$, then equation (3.5) says that the minimal cost $\langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$ equals the "cost until now" $\int_0^t \|y(t)\|_{\mathtt{Y}}^2\,dt$ plus the minimal cost over the remaining time interval $[t,\infty)$. The latter cost equals the minimal cost $\langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}}$ with initial state $x(t)$. This is often called the principle of dynamic programming (or the principle of optimization). By (3.5), it also applies to the indefinite case (to general $J = J^* \in \mathcal{B}(\mathtt{Y})$).

Similarly, (3.7) says that while $\int_0^t \|y(t)\|_{\mathtt{Y}}^2\,dt$ converges to the minimal cost $\int_0^\infty \|y(t)\|_{\mathtt{Y}}^2\,dt$, the remaining cost $\langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}}$ converges to zero.

One could derive from (3.5) and (3.8) that $\langle x_0, \mathcal{P}\tilde{x}_0\rangle_{\mathtt{X}} = \langle x(t), \mathcal{P}\tilde{x}(t)\rangle_{\mathtt{X}} + \langle y, \pi_{[0,t)}J\tilde{y}\rangle_{\mathrm{L}^2}$ when u is J-optimal for x_0 and $\tilde{u} \in \mathcal{U}(\tilde{x}_0)$ (and $\tilde{x} := \mathscr{A}\tilde{x}_0 + \mathscr{B}\tau\tilde{u}$, $\tilde{y} := \mathscr{C}\tilde{x}_0 + \mathscr{D}\tilde{u}$); this is an "indefinite form" of (3.5). Equation (3.8) is the special case of this with $\tilde{x}_0 = 0$, $\tilde{u} = \eta$. Similarly, (3.9) says that $\langle x(t), \mathcal{P}\tilde{x}(t)\rangle_{\mathtt{X}} \to 0$; it is equivalent to the orthogonality condition $\langle y, J\mathscr{D}\eta\rangle_{\mathrm{L}^2} = 0$ (under (3.8)).

Equation (3.11) follows from (3.5) by partial integration (through (3.19)); the proof of (d) is analogous. Equations (3.10) and (3.11) are actually exactly the equations (3.8) and (3.5) for the system Σ_+ (and $J_{\mathcal{P}}$) introduced in Lemma 3.9 below; see its proof.

Proof of Lemma 3.8*:* (a) To obtain (a), expand the equality $\langle y_1 + y_0, J(y_1 + y_0)\rangle = \langle x_1 + x_0, \mathcal{P}(x_1 + x_0)\rangle$, where $y_k := \mathscr{C}x_k + \mathscr{D}u_k$ $(k = 0, 1)$, and then replace x_0 by ix_0.

(b) 1° Let $\eta \in \mathcal{U}(0)$. Then $\tau^{-t}\eta \in \mathcal{U}(0)$, by Lemma 3.6, hence

$$0 = \langle Jy, \mathscr{D}\tau^{-t}\eta\rangle = \langle Jy, \pi_+\tau^{-t}\mathscr{D}\eta\rangle. \tag{3.12}$$

Since $\pi_{[t,\infty)} = \pi_+ - \pi_{[0,t)}$, it follows that

$$\langle J\pi_+\tau^t y, \mathscr{D}\eta\rangle = \langle Jy, \tau^{-t}\pi_+\mathscr{D}\eta\rangle = \langle Jy, \pi_{[t,\infty)}\tau^{-t}\mathscr{D}\eta\rangle = -\langle Jy, \pi_{[0,t)}\tau^{-t}\mathscr{D}\eta\rangle = 0, \tag{3.13}$$

because $\pi_{[0,t)}\tau^{-t}\mathscr{D}\eta = \tau^{-t}\pi_{[-t,0)}\mathscr{D}\eta = 0$ (since $\pi_-\mathscr{D}\pi_+ = 0$). By (2.4), equation (3.13) says that $\pi_+\tau^t u$ is J-optimal for $x(t)$, hence

$$\langle x(t), \mathcal{P}x(t)\rangle_{\mathtt{X}} = \mathcal{J}(x(t), \pi_+\tau^t u) = \langle \pi_+\tau^t y, J\pi_+\tau^t y\rangle = \langle y, \pi_{[t,\infty)}Jy\rangle \tag{3.14}$$

(use Definition 3.7). This proves (3.7) and (3.5)

2° Claims (3.8)–(3.11) follow from (c), (d) and (e) (whose proofs only use 1°). Let $t \to +\infty$ in (3.11) to obtain (3.6) (case $r = 0$ is trivial; for $r > 0$ we can use the fact that $\mathrm{e}^{-r\cdot}x \to 0$, because Σ is $r - \epsilon$ stable and $\mathrm{e}^{-rt}\|\tau^t u\|_{\mathrm{L}^2_{r-\epsilon}} = \mathrm{e}^{-\epsilon t}\|u\|_{\mathrm{L}^2_{r-\epsilon}} \to 0$).

(c) By letting $t \to +\infty$ in (3.8) for any $\eta \in \mathcal{U}(0)$, we get $0 = -\langle y, J\mathscr{D}\eta\rangle$, hence "if" holds. Assume then that u is J-optimal. Given $\eta \in \mathrm{L}^2((0,t);\mathtt{U})$, extend it by setting $\tilde{\eta} := \pi_{[0,t)}\eta + \tau^{-t}\tilde{u}_{\mathrm{opt}}$ for some J-optimal $\tilde{u}_{\mathrm{opt}}$ for $\mathscr{B}^t\eta$. By Lemma 3.6, $\tilde{\eta} \in \mathcal{U}(0)$. By (2.4), $\pi_+\tau^t\mathscr{D}\tilde{\eta} = \mathscr{C}\mathscr{B}^t\eta + \mathscr{D}^t\tilde{u}_{\mathrm{opt}} =: \tilde{y}_{\mathrm{opt}}$; by (b), $\pi_+\tau^t u$ is J-optimal for $x(t)$. Therefore,

$$\begin{aligned} 0 &= \langle Jy, \mathscr{D}\tilde{\eta}\rangle = \langle(\pi_{[0,t)} + \tau^{-t}\tau^t\pi_{[t,\infty)})Jy, \mathscr{D}\tilde{\eta}\rangle && (3.15)\\ &= \langle \pi_{[0,t)}Jy, \mathscr{D}\tilde{\eta}\rangle + \langle J\pi_+\tau^t y, \mathscr{D}\tau^t\tilde{\eta}\rangle && (3.16)\\ &= \langle y, J\mathscr{D}^t\tilde{\eta}\rangle + \langle \mathscr{C}x(t) + \mathscr{D}\pi_+\tau^t u, J\tilde{y}_{\text{opt}}\rangle && (3.17)\\ &= \langle y, J\mathscr{D}^t\eta\rangle + \langle x(t), \mathcal{P}\mathscr{B}^t\eta\rangle, && (3.18)\end{aligned}$$

where the last equality is from (a). Thus, (3.8) holds (for any $\eta \in \mathrm{L}^2([0,t);\mathtt{U})$, hence for any $\eta \in \mathrm{L}^2_{\text{loc}}(\mathbb{R}_+;\mathtt{U})$, because (3.8) depends on $\eta_{|[0,t)}$ only).

Let $t \to +\infty$ to obtain (3.9) (because $\langle y, J\mathscr{D}\eta\rangle = 0$).

(d) The proof is analogous to that of (e) (but simpler) and hence omitted.

(e) If (3.5) holds, then $\langle x, \mathcal{P}x\rangle_{\mathtt{X}} = \langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}} + \int_0^t \langle y, Jy\rangle_{\mathtt{Y}}(t)\,dt$, hence then

$$\langle x, \mathcal{P}x\rangle_{\mathtt{X}} \in \mathrm{AC}_{\text{loc}} \ \& \ \langle x, \mathcal{P}x\rangle_{\mathtt{X}}'(t) = -\langle y(t), Jy(t)\rangle_{\mathtt{Y}} \quad \text{for a.e. } t \geq 0. \qquad (3.19)$$

(Here AC_{loc} stands for locally absolutely continuous functions.) Conversely, if (3.19) holds, so does (3.5) too, because its both sides are equal for $t = 0$.

Similarly, if (3.11) holds, then the facts that

$$(1 - \mathrm{e}^{-2rt})\langle x, \mathcal{P}x\rangle_{\mathtt{X}}(t) = \langle \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right], \pi_{[0,t)} \left[\begin{smallmatrix} J & 0 \\ 0 & 2r\mathcal{P} \end{smallmatrix}\right] \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]\rangle_{\mathrm{L}^2_r} - \langle y, \pi_{[0,t)}Jy\rangle_{\mathrm{L}^2} =: f(t) \quad (3.20)$$

and $f \in \mathrm{AC}_{\text{loc}}$ imply that

$$\langle x, \mathcal{P}x\rangle_{\mathtt{X}} \in \mathrm{AC}_{\text{loc}} \quad \text{and} \quad \langle x, \mathcal{P}x\rangle_{\mathtt{X}}'(t) + 2r\mathrm{e}^{-2rt}\langle x, \mathcal{P}x\rangle_{\mathtt{X}}(t) = f'(t) \quad \text{a.e.} \qquad (3.21)$$

(on $(0,\infty)$, hence on $[0,\infty)$, because x is continuous (hence $\langle x, \mathcal{P}x\rangle_{\mathtt{X}}$ too) and $f' - 2r\mathrm{e}^{-2r\cdot}\langle x, \mathcal{P}x\rangle_{\mathtt{X}} \in \mathrm{L}^1([0,\infty))$). Conversely, if (3.19) holds, then the derivatives of both sides of (3.11) are equal a.e. □

The optimal control problem has already been solved for stable systems. Therefore, we want to replace A by $A-\alpha$ to make the system exponentially stable. To retain the same J-optimal cost operator $\mathcal{P}$, we must add the cost $2r\langle x, \mathcal{P}x\rangle_{\mathrm{L}^2}$:

Lemma 3.9. *Let $\alpha \in \mathbb{C}$ be such that $0 \leq r := \operatorname{Re}\alpha > \omega_A$. Then the system*

$$\Sigma_+ := \left[\begin{array}{c|c} \mathscr{A}_+ & \mathscr{B}_+ \\ \hline \mathscr{C}_+ & \mathscr{D}_+ \end{array}\right] := \left[\begin{array}{c|c} \mathrm{e}^{-\alpha\cdot}\mathscr{A} & \mathscr{B}\mathrm{e}^{\alpha\cdot} \\ \hline \mathrm{e}^{-\alpha\cdot}\mathscr{C} & \mathrm{e}^{-\alpha\cdot}\mathscr{D}\mathrm{e}^{\alpha\cdot} \\ \mathrm{e}^{-\alpha\cdot}\mathscr{A} & \mathrm{e}^{-\alpha\cdot}\mathscr{B}\tau\mathrm{e}^{\alpha\cdot} \end{array}\right], \qquad (3.22)$$

is an exponentially stable WPLS on $(\mathtt{U}, \mathtt{X}, \mathtt{Y} \times \mathtt{X})$.

Assume that $\mathcal{P}$ is the J-optimal cost operator for Σ and J. Then $\mathcal{P}$ is the $J_{\mathcal{P}}$-optimal cost operator for Σ_+, where $J_{\mathcal{P}} := \left[\begin{smallmatrix} J & 0 \\ 0 & 2r\mathcal{P} \end{smallmatrix}\right]$, and if $x_0 \in \mathtt{X}$ and $u \in \mathcal{U}(x_0)$ is J-optimal, then $\mathrm{e}^{-\alpha\cdot}u$ is $J_{\mathcal{P}}$-optimal for Σ_+.

Proof. The first claim is from [Sta05, Example 2.3.5] or [Mik02, Remark 6.1.9]. Assume that $x_0 \in \mathtt{X}$ and that u is a J-optimal control. Set $u_+ := \mathrm{e}^{-\alpha\cdot}u$. We have

$$\mathrm{e}^{\alpha\cdot}\tau^T u = \mathrm{e}^{\alpha\cdot}u(\cdot + T) = \mathrm{e}^{-\alpha T}\mathrm{e}^{\alpha(\cdot+T)}u(\cdot + T) = \mathrm{e}^{-\alpha T}\tau^T(\mathrm{e}^{\alpha\cdot}u), \qquad (3.23)$$

hence $\mathscr{B}_+\tau^T u = \mathrm{e}^{-\alpha T}\mathscr{B}\tau^T\mathrm{e}^{\alpha\cdot}u$, i.e., $\mathscr{B}_+\tau = \mathrm{e}^{-\alpha\cdot}\mathscr{B}\tau\mathrm{e}^{\alpha\cdot}$. Consequently,

$$x_+ := \mathscr{A}_+x_0 + \mathscr{B}_+\tau u_+ = \mathrm{e}^{-\alpha\cdot}x \quad \text{and} \quad y_+ := \mathscr{C}_+x_0 + \mathscr{D}_+u_+ = \mathrm{e}^{-\alpha\cdot}\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]. \tag{3.24}$$

Therefore, (3.10) equals (3.8) with Σ_+ in place of Σ and $J_{\mathcal{P}}$ in place of J (and $\mathrm{e}^{-\alpha\cdot}\eta$ in place of η). By (c1) and (c2), it follows that u_+ is $J_{\mathcal{P}}$-optimal for x_0 and Σ_+.

By (3.24), we have $\langle y_+, J_{\mathcal{P}}y_+\rangle_{\mathrm{L}^2} = \langle y, Jy\rangle_{L^2_r} + \langle x, 2r\mathcal{P}x\rangle_{L^2_r}$. But, by (3.6), this equals $\langle y, Jy\rangle_{L^2} = \langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$. Since x_0 was arbitrary and u_+ was $J_{\mathcal{P}}$-optimal, the operator $\mathcal{P}$ is the $J_{\mathcal{P}}$-optimal cost operator for Σ_+. □

4. Minimizing state feedback

In this section we deduce certain properties of minimizing state feedback.

An admissible state-feedback pair $[\ \mathscr{F} \mid \mathscr{G}\ ]$ for Σ is called *J-optimal* (resp. *$\mathcal{J}$-minimizing*) if for any $x_0 \in \mathtt{X}$ the control $\mathscr{F}_\circlearrowleft x_0$ is J-optimal (resp. $\mathcal{J}$-minimizing) for x_0.

Now we can establish our main result:

Lemma 4.1. *If the output-FCC* (1.5) *holds, then there exists a* (1.10)*-minimizing state-feedback pair* $[\ \mathscr{F} \mid \mathscr{G}\ ]$ *for* Σ. *The pair is unique modulo* (A.6).

Proof. By Lemmata 3.2 and 3.4, there exists an I-optimal cost operator $\mathcal{P}$ for $\tilde{\Sigma}$. Fix some α as in Lemma 3.9; then $\mathcal{P}$ is $J_{\mathcal{P}}$-optimal for $\tilde{\Sigma}_+$ (which is defined by (3.22) with $\tilde{\Sigma}$ in place of Σ).

The output of $\tilde{\Sigma}_+$ contains a copy of the input (because $\begin{bmatrix} 0 & I \end{bmatrix}\mathrm{e}^{-\alpha\cdot}\tilde{\mathscr{D}}\mathrm{e}^{\alpha\cdot} = I$). Therefore, the system is "$J_{\mathcal{P}}$-coercive" in terms of [Sta98d]. Consequently, [Sta98d, Lemma 2.5 & Theorem 2.6(i)] imply that there exists a $J_{\mathcal{P}}$-optimal state-feedback pair $[\ \mathscr{F}_+ \mid \mathscr{G}_+\ ]$ for $\tilde{\Sigma}_+$ and that the $J_{\mathcal{P}}$-optimal control for $\tilde{\Sigma}_+$ is unique for every $x_0 \in \mathtt{X}$.

It easily follows that $[\ \mathscr{F} \mid \mathscr{G}\ ]$ is an admissible state-feedback pair for $\tilde{\Sigma}$ (hence for Σ too), where $\mathscr{F} := \mathrm{e}^{\cdot\alpha}\mathscr{F}_+$, $\mathscr{G} := \mathrm{e}^{\cdot\alpha}\mathscr{G}_+\mathrm{e}^{-\cdot\alpha}$, $\mathscr{F}_\circlearrowleft = \mathrm{e}^{\cdot\alpha}\mathscr{F}_{+\circlearrowleft}$ and $\mathscr{F}_{+\circlearrowleft} := (I - \mathscr{G}_+)^{-1}\mathscr{F}_+$ [Mik02, Remark 6.1.9].

By uniqueness and Lemma 3.9, the control $\mathscr{F}_\circlearrowleft x_0 = \mathrm{e}^{-\alpha\cdot}\mathscr{F}_{+\circlearrowleft}x_0$ must equal $u_{\mathrm{opt}}^{x_0}$ for any $x_0 \in \mathtt{X}$, hence $[\ \mathscr{F} \mid \mathscr{G}\ ]$ is I-optimal for $\tilde{\Sigma}$, i.e., (1.10)-minimizing for Σ, and $\mathscr{F}_\circlearrowleft$ is unique.

Since $\mathscr{F}_\circlearrowleft$ is unique, the pair $[\ \mathscr{F} \mid \mathscr{G}\ ]$ is unique modulo (A.6), by Lemma A.5. □

We could deduce Theorems 1.2 and 1.1 from Lemma 4.1 (and the fact that $\mathscr{F}_\circlearrowleft x_0 \in \mathcal{U}(x_0)$), but to avoid unnecessary details, we first establish one more useful Riccati equation, which we anyway need for Theorem 1.3.

To any J-optimal state-feedback pair corresponds a unique *signature* operator S:

Lemma 4.2. *Let* $[\,\mathscr{F} \mid \mathscr{G}\,]$ *be a* J*-optimal state-feedback pair for* Σ*. Then* $\mathscr{C}_\circlearrowleft$ *and* $\mathscr{F}_\circlearrowleft$ *are stable,* $\mathcal{P} = \mathscr{C}_\circlearrowleft^* J \mathscr{C}_\circlearrowleft$ *is the* J*-optimal cost operator and there exists* $S = S^* \in \mathcal{B}(\mathtt{U})$ *such that for each* $t \geq 0$ *we have*

$$\pi_{[0,t)} S = \mathscr{N}^{t*} J \mathscr{N}^t + \mathscr{B}_\circlearrowleft^{t\,*} \mathcal{P} \mathscr{B}_\circlearrowleft^t. \tag{4.1}$$

If $J = I$*, then* $\mathcal{P}, S \geq 0$*,* $\mathscr{N}$ *is stable,* $\mathscr{N}^* \mathscr{C}_\circlearrowleft = 0$ *and* $\mathscr{N}^* \mathscr{N} = S$*.*

Recall from (2.6) that $\mathscr{M} := (I - \mathscr{G})^{-1} = \mathscr{G}_\circlearrowleft + I$ and $\mathscr{N} := \mathscr{D}\mathscr{M} = \mathscr{D}_\circlearrowleft$. Note that we identify $S \in \mathcal{B}(\mathtt{U})$ with the multiplication operator $u \mapsto Su$.

Proof. 1° For $u = \mathscr{F}_\circlearrowleft x_0$ we have $y = \mathscr{C} x_0 + \mathscr{D} u = \mathscr{C}_\circlearrowleft x_0$ and for any $x_0 \in \mathtt{X}$. But $\mathscr{F}_\circlearrowleft x_0 \in \mathcal{U}(x_0)$, hence $u, y \in \mathrm{L}^2$. Since $x_0 \in \mathtt{X}$ was arbitrary, the maps $\mathscr{C}_\circlearrowleft$ and $\mathscr{F}_\circlearrowleft$ are stable, by Lemma A.1. But $\mathcal{J}(x_0, u) = \langle y, Jy \rangle_{\mathrm{L}^2} = \langle x_0, \mathscr{C}_\circlearrowleft^* J \mathscr{C}_\circlearrowleft x_0 \rangle_{\mathtt{X}}$, for every $x_0 \in \mathtt{X}$, hence $\mathcal{P} = \mathscr{C}_\circlearrowleft^* J \mathscr{C}_\circlearrowleft$.

By Lemma A.2 (with $\Sigma_\circlearrowleft$ in place of Σ), we have $\left[\begin{smallmatrix} \mathscr{D} \\ I \end{smallmatrix}\right] \mathscr{M}[\mathrm{L}^2_{\mathrm{c}}] = \left[\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}\right] [\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2$, hence $\mathscr{M}[\mathrm{L}^2_{\mathrm{c}}(\mathbb{R}_+; \mathtt{U})] \subset \mathcal{U}(0)$. Consequently, J-optimality implies that

$$\langle \mathscr{N} \eta_\circlearrowleft, J \mathscr{C}_\circlearrowleft x_0 \rangle_{\mathrm{L}^2} = \langle \mathscr{D}\mathscr{M} \eta_\circlearrowleft, J \mathscr{C}_\circlearrowleft x_0 \rangle_{\mathrm{L}^2} = 0 \quad \text{for every} \quad \eta_\circlearrowleft \in \mathrm{L}^2_{\mathrm{c}}(\mathbb{R}_+; \mathtt{U}). \tag{4.2}$$

By 4. of Definition 2.1, $\langle \mathscr{N} \pi_+ v, J \mathscr{N} \pi_- u \rangle = \langle \mathscr{N} \pi_+ v, J \mathscr{C}_\circlearrowleft \mathscr{B}_\circlearrowleft u \rangle = 0$ for all $u, v \in \mathrm{L}^2_{\mathrm{c}}$. By Lemma A.4, it follows that there exists a unique $S = S^* \in \mathcal{B}(U)$ such that $\langle \mathscr{N} v, J \mathscr{N} u \rangle = \langle v, Su \rangle$ $(u, v \in \mathrm{L}^2_{\mathrm{c}})$. This implies that

$$\pi_{[0,t)} \mathscr{N}^* (\pi_+) J \mathscr{N} \pi_{[0,t)} = \pi_{[0,t)} S \qquad (t \geq 0) \tag{4.3}$$

(where π_+ is redundant). From the identities $\mathcal{P} = \mathscr{C}_\circlearrowleft^* J \mathscr{C}_\circlearrowleft$ and

$$\pi_{[t,\infty)} \mathscr{N} \pi_{[0,t)} = \pi_{[t,\infty)} \tau^{-t} \mathscr{N} \tau^t \pi_{[0,t)} = \tau^{-t} \pi_+ \mathscr{N} (\pi_-) \tau^t \pi_{[0,t)} = \tau^{-t} \mathscr{C}_\circlearrowleft \mathscr{B}_\circlearrowleft \tau^t \pi_{[0,t)} \tag{4.4}$$

it follows that

$$\mathscr{B}_\circlearrowleft^{t\,*} \mathcal{P} \mathscr{B}_\circlearrowleft^t = (\pi_{[t,\infty)} \mathscr{N} \pi_{[0,t)})^* J \pi_{[t,\infty)} \mathscr{N} \pi_{[0,t)} = \pi_{[0,t)} \mathscr{N}^* (\pi_+ - \pi_{[0,t)}) J \mathscr{N} \pi_{[0,t)}. \tag{4.5}$$

Combine this with (4.3) to observe that (4.1) holds for any $t \geq 0$.

2° Assume that $J = I$. Then $\mathcal{P}, S \geq 0$, by (4.3) and the fact that $\mathcal{P} = \mathscr{C}_\circlearrowleft^* \mathscr{C}_\circlearrowleft$. From (4.3) we observe that $\|\mathscr{N} \pi_{[0,t)} u\|_2^2 = \|S^{1/2} \pi_{[0,t)} u\|_2^2$ $(u \in \mathrm{L}^2)$. Letting $t \to +\infty$, we observe that $\|\mathscr{N} u\|_2^2 = \|S^{1/2} u\|_2^2 < \infty$ $(u \in \mathrm{L}^2)$, by the Monotone Convergence Theorem, i.e., $\mathscr{N}^* \mathscr{N} = S$. From (4.2) we get that $\mathscr{N}^* J \mathscr{C}_\circlearrowleft = 0$. □

In the case $J = I$ (or $J \geq \epsilon I$), we obtain from $y = \mathscr{C}_\circlearrowleft x_0 + \mathscr{N} u_\circlearrowleft$ and the above that

$$\mathcal{J} = \langle y, Jy \rangle_{\mathrm{L}^2} = \langle x_0, \mathcal{P} x_0 \rangle_{\mathtt{X}} + \langle u_\circlearrowleft, S u_\circlearrowleft \rangle_{\mathrm{L}^2}. \tag{4.6}$$

Thus, the J-optimal cost is then particularly robust with respect to any external disturbance $u_\circlearrowleft \in \mathrm{L}^2(\mathbb{R}_+; \mathtt{U})$ in the state-feedback loop.

Given a nonnegative cost function (e.g., $J \geq 0$), the minimal cost $\langle x(t), \mathcal{P} x(t) \rangle$ vanishes to zero, for any admissible u:

Lemma 4.3. *Let* $[\ \mathscr{F} \mid \mathscr{G}\ ]$ *be a* J*-optimal state-feedback pair for* Σ *and assume that* $\mathcal{J}(\cdot,\cdot) \ge 0$*. Then* $\langle \mathscr{B}^t u, \mathcal{P}\mathscr{B}^t u\rangle_{\mathtt{X}} \to 0$*, as* $t \to +\infty$*, for each* $u \in \mathcal{U}(0)$*.*

If $S \ge \epsilon I$*,* $\epsilon > 0$*, then* $\mathscr{M}^{-1}[\mathcal{U}(0)] \subset \mathrm{L}^2(\mathbb{R}_+;U)$*.*

Proof. From Lemma 3.3 we observe that $u := \mathscr{F}_{\circlearrowleft} x_0$ minimizes $\mathcal{J}(x_0,\cdot)$. The minimal cost $\langle x_0, \mathcal{P}x_0\rangle_{\mathtt{X}}$ is nonnegative, i.e., $\mathcal{P} \ge 0$.

Given $u \in \mathcal{U}(0)$ and $t \ge 0$, set $x_t := \mathscr{B}^t u$, $\tilde{u} := \pi_+\tau^t u \in \mathcal{U}(x_t)$ (Lemma 3.6). Since $\pi_+\tau^t = \tau^t\pi_{[t,\infty)}$, we get from (2.4) that

$$\langle \mathscr{C}x_t + \mathscr{D}\tilde{u}, J\mathscr{C}x_t + \mathscr{D}\tilde{u}\rangle = \langle \pi_+\tau^t\mathscr{D}u, J\pi_+\tau^t\mathscr{D}u\rangle = \langle \pi_{[t,\infty)}\mathscr{D}u, J\pi_{[t,\infty)}\mathscr{D}u\rangle. \quad (4.7)$$

But $\langle x_t, \mathcal{P}x_t\rangle_{\mathtt{X}}$ is the minimum of $\langle \mathscr{C}x_t + \mathscr{D}\tilde{u}, J\mathscr{C}x_t + \mathscr{D}\tilde{u}\rangle$ over $\tilde{u} \in \mathcal{U}(x_t)$. By (4.7), this implies that $\langle x_t, \mathcal{P}x_t\rangle_{\mathtt{X}} \le \langle \mathscr{D}u, \pi_{[t,\infty)}J\mathscr{D}u\rangle$. But $\langle \mathscr{D}u, \pi_{[t,\infty)}J\mathscr{D}u\rangle \to 0$, as $t \to +\infty$, hence $\langle x_t, \mathcal{P}x_t\rangle_{\mathtt{X}} \to 0$, as claimed.

Set now $v := \mathscr{M}^{-1}u$ to obtain from (4.1) and (2.6) that

$$\langle v, S\pi_{[0,t)}v\rangle_{\mathrm{L}^2} = \langle \mathscr{D}^t u, J\mathscr{D}^t u\rangle_{\mathrm{L}^2} + \langle \mathscr{B}^t u, \mathcal{P}\mathscr{B}^t u\rangle_{\mathtt{X}}, \quad (4.8)$$

hence $\lim_{t\to+\infty} \langle v, S\pi_{[0,t)}v\rangle_{\mathrm{L}^2} = \langle \mathscr{D}u, J\mathscr{D}u\rangle = \mathcal{J}(0,u)$. If $S \ge \epsilon I$, $\epsilon > 0$, then this implies that $v \in \mathrm{L}^2$. □

Maps $\mathscr{M} \in \mathrm{TIC}_0(\mathtt{U})$ and $\mathscr{N} \in \mathrm{TIC}_0(\mathtt{U},\mathtt{Y})$ are called *quasi–right coprime* if

$$u \in \mathrm{L}^2 \Leftrightarrow \begin{bmatrix}\mathscr{N}\\ \mathscr{M}\end{bmatrix} u \in \mathrm{L}^2 \quad \text{for every} \quad u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U}) \ \text{ and } \ \omega \in \mathbb{R} \quad (4.9)$$

(this actually holds for every $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathtt{U})$ when, in addition, $\mathscr{M} \in \mathcal{G}\mathrm{TIC}_\infty$; see [Mik02] for details).

By Lemma 4.2, the (1.10)-minimizing state-feedback pair is output-stabilizing for Σ. Now we show that also $\mathscr{N}$ and $\mathscr{M}$ become stable and quasi–right coprime:

Lemma 4.4. *Assume the output-FCC* (1.5)*. Then any* (1.10)*-minimizing state-feedback pair* $[\ \mathscr{F} \mid \mathscr{G}\ ]$ *is output- and I/O-stabilizing,* $\mathscr{N}$ *and* $\mathscr{M}$ *are quasi–right coprime, and* $S := \mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} \in \mathcal{GB}(\mathtt{U})$*. Moreover,* $[\ \mathscr{F} \mid \mathscr{G}\ ]$ *can be chosen so that* $S = I$ *and* $\mathscr{M}$ *is an isometric isomorphism of* $\mathrm{L}^2(\mathbb{R}_+;U)$ *onto* $\mathcal{U}(0)$*, where* $\|u\|^2_{\mathcal{U}(0)} := \|u\|_2^2 + \|\mathscr{D}u\|_2^2$*.*

The last property of the lemma can be used to reduce unstable problems to the stable case by preliminary state feedback [Mik05b].

Proof. 1° Set $\tilde{\mathscr{N}} := \tilde{\mathscr{D}}\mathscr{M}$, where $\mathscr{M} := (I - \mathscr{G})^{-1}$. By Lemma 4.2 (applied to $\tilde{\Sigma}$ and I), we have $\tilde{\mathcal{P}} \ge 0$, $\tilde{\mathscr{N}} \in \mathrm{TIC}_0$, $\tilde{S} := \tilde{\mathscr{N}}^*\tilde{\mathscr{N}} \in \mathcal{B}(\mathtt{U})$. But

$$\tilde{\mathscr{N}} := \tilde{\mathscr{D}}\mathscr{M} = \begin{bmatrix}\mathscr{D}\\ I\end{bmatrix}\mathscr{M} = \begin{bmatrix}\mathscr{N}\\ \mathscr{M}\end{bmatrix}, \quad (4.10)$$

hence $\mathscr{N}, \mathscr{M} \in \mathrm{TIC}_0$ (i.e., $[\ \mathscr{F} \mid \mathscr{G}\ ]$ is I/O-stabilizing for Σ) and $\tilde{S} = \mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} =: S$. Fix some $t > 0$. By (4.1), $\pi_{[0,t)}S \ge (\tilde{\mathscr{N}}^t)^*\tilde{\mathscr{N}}^t \ge \mathscr{M}^{t*}\mathscr{M}^t$. But

$$\pi_{[0,t)}\mathscr{M}^{-1}\pi_{[0,t)}\mathscr{M}\pi_{[0,t)} = \pi_{[0,t)} \quad (4.11)$$

(because $\pi_- \mathscr{M}^{-1}\pi_+ = 0$ implies that $\pi_{[0,t)}\mathscr{M}^{-1}\pi_{[t,\infty)} = 0$), hence $\mathscr{M}^{t*}\mathscr{M}^t \geq \epsilon\pi_{[0,t)}$ for some $\epsilon > 0$. From $S\pi_{[0,t)} \geq \epsilon\pi_{[0,t)}$ we conclude that $S \geq \epsilon I$. By Lemma A.5 (set $Q := S^{1/2}$), we can redefine $[\, \mathscr{F} \mid \mathscr{G} \,]$ so that it is remains J-optimal (and output- and I/O-stabilizing, by (2.6)) but $\mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} = S^{-1/2}SS^{-1/2} = I$.

2° Consequently, $[\begin{smallmatrix} \mathscr{Q} \\ I \end{smallmatrix}]\mathscr{M}u = [\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}]u$ has the norm $\|u\|_2$, for any $u \in \mathrm{L}^2(\mathbb{R};\mathtt{U})$. In particular, $\mathscr{M}$ maps $\mathrm{L}^2(\mathbb{R}_+;\mathtt{U}) \to \mathcal{U}(0)$, isometrically. By Lemma 4.3, $\mathscr{M}$ is onto.

3° If $u \in \mathrm{L}^2_\omega(\mathbb{R}_+;\mathtt{U})$, $\omega \in \mathbb{R}$ and $\mathrm{L}^2 \ni [\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}]u = [\begin{smallmatrix} \mathscr{Q} \\ I \end{smallmatrix}]\mathscr{M}u$, then $\mathscr{M}u \in \mathcal{U}(0)$, hence $u \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{U})$, by Lemma 4.3. Thus, $\mathscr{N}$ and $\mathscr{M}$ are quasi–right coprime (hence so are the original ones, $\mathscr{N}Q$ and $\mathscr{M}Q$). □

5. Main results

In this section we shall present direct generalizations of Theorems 1.1, 1.2 and 1.3 and Corollary 1.4 to arbitrary WPLSs.

From Lemmata 4.1 and 4.4 we obtain the following generalization of Theorem 1.3:

Corollary 5.1. *If the output-FCC holds, then there exists an output- and I/O-stabilizing state-feedback pair for which* $\mathscr{N}$ *and* $\mathscr{M}$ *are quasi–right coprime and* $\mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} = I$. □

(The latter condition is usually expressed by calling $[\begin{smallmatrix} \mathscr{N} \\ \mathscr{M} \end{smallmatrix}]$ inner or normalized.)

This generalizes also Corollary 1.4:

Corollary 5.2. *Any function having an output-stabilizable realization has a quasi–right-coprime factorization.* □

This means that then $\hat{\mathscr{D}} = \hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$, where $\mathscr{N}$ and $\mathscr{M}$ are quasi–right coprime and $\mathscr{M}^{-1}$ is proper. [Mik02]

From Corollary 5.1 we also get the generalization of Theorem 1.2:

Corollary 5.3. *The output-FCC* (1.5) *holds iff the system is output-stabilizable.* □

Indeed, the necessity of (1.5) is obvious, as noted in the introduction.

By setting above $C = I$ and $D = 0$, we get the generalization of Theorem 1.1:

Corollary 5.4. *The state-FCC* (1.4) *holds iff the system is exponentially stabilizable.* □

Proof. "If" is again obvious, so assume that Σ satisfies the state-FCC (1.4), or equivalently, that $\tilde{\Sigma} := [\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{A} & \mathscr{B}\tau \end{smallmatrix}]$ does. If $[\, \mathscr{F} \mid \mathscr{G} \,]$ is an output-stabilizing state-feedback pair for $\tilde{\Sigma}$ with closed-loop system $\tilde{\Sigma}_\circlearrowleft$, then $[\, \mathscr{F} \mid \mathscr{G} \,]$ is admissible for Σ too and $\tilde{\mathscr{C}}_\circlearrowleft = \mathscr{A} + \mathscr{B}\tau\mathscr{F}_\circlearrowleft = \tilde{\mathscr{A}}_\circlearrowleft = \mathscr{A}_\circlearrowleft$. But the stability of $\tilde{\mathscr{C}}_\circlearrowleft$ means that $\tilde{\mathscr{C}}_\circlearrowleft x_0 \in \mathrm{L}^2(\mathbb{R}_+;\mathtt{X})$ for all $x_0 \in \mathtt{X}$, hence $\Sigma_\circlearrowleft$ is exponentially stable, by Lemma 2.2 (since $\mathscr{A}_\circlearrowleft = \tilde{\mathscr{C}}_\circlearrowleft$). □

A trivial consequence of Corollary 5.4 is that a system is estimatable iff it is exponentially detectable (these are the dual properties of optimizability and exponential detectability; see [WR01], [Sta98a] or [Mik02] for details).

6. Notes

The *Linear Quadratic Regulator (LQR)* problem amounts to the minimization of (1.10) by state feedback. It has been used both for optimizing certain systems and for finding a stabilizing state feedback.

We have shown above that the minimizing control is given by an admissible state feedback and then used this result to establish for general WPLSs the results mentioned in the introduction. We discuss below some earlier results in the same direction and some future extensions of our results, including constructive formulas for the J-optimal cost operator $\mathcal{P}$ and the state feedback.

The classical state-feedback solution of the LQR problem can easily be extended to infinite-dimensional systems having bounded B and C, as shown in [CZ95], which also contains further historical remarks on the problem (p. 333).

A solution of the LQR problem for WPLSs (without admissible state feedback) was given in [Zwa96], which more or less contains Lemmata 3.1, 3.2 and 3.6. Zwart applied directly the orthogonality of the optimal control. Similar results for stable WPLSs with the general cost function (3.2) were given in [Sta98b], which uses Fréchet derivatives. In [Sta98b] it was also shown that the optimal control can be given by state feedback if the Popov function has a spectral factorization.

Lemmata 3.3 and 3.5 are from [Mik02] and Lemmata 3.8 and 3.9 seem to be new, as well as the results of Section 4 except that the normalization $\mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} = I$ is a well-known consequence of spectral factorization. The results of Appendix A are from [Mik02].

Some kind of integral Riccati equations have been used for decades (see, e.g., [CP78]) and for stable or jointly stabilizable and detectable WPLSs certain special cases of (3.5) and (3.8) were given in [Sta98d] (including (4.1) and (4.6)) and extended to arbitrary WPLSs in [Mik02], where also the converse implication was shown: if a (stabilizing) solution to the Riccati equations exists, then the corresponding state-feedback pair produces the (J-)optimal control. In [Mik05a] a generalization of those results shall be presented, including necessary and sufficient "Integral Riccati equation" conditions for optimal control or state feedback, in both time-domain and frequency-domain terms.

In practical applications, the transfer function $\hat{\mathscr{D}}$ has a (weak) limit $D := \lim_{s\to+\infty} \hat{\mathscr{D}}(s)$ at infinity along the positive real axis. Such systems are called *(weakly) regular*, and for them the *algebraic Riccati equation (ARE)* can be defined and shown to be equivalent to the integral one (provided that also $\hat{\mathscr{F}}(+\infty)$ exists; moreover, the left-hand-side of (1.9) must be modified if B is highly unbounded). In the (regular) WPLS setting, the necessity of the AREs was first established in

[Sta97] and [WW97], independently (in the stable case; the sufficiency and general case in [Mik02]).

However, the unboundedness of B and C makes the AREs rather complicated. Therefore, the *reciprocal AREs* were introduced in [Cur03]. They are non-standard AREs but can be shown equivalent to the original ones and they use bounded coefficients only, so they allow one to reduce several proofs and equations to a simple special case. See [OC04] for the reciprocal ARE corresponding to (1.10). By the proof of Corollary 5.1, their AREs define an admissible state feedback pair (and that pair is output- and I/O-stabilizing and makes $\mathscr{N}$ and $\mathscr{M}$ quasi–right coprime) iff a solution exists. Their method requires that $i\mathbb{R}\cap\rho(A)$ is nonempty, but that can be circumvented by using the proof Lemma 3.9 to obtain the reciprocal AREs for Σ_+. We shall provide the details in [Mik05c], where also the general cost function (3.2) will be considered. In fact, the analysis of the optimal control problem corresponding to such AREs was what lead to the methods of Lemmata 3.8 and 3.9.

We have above studied the set $\mathcal{U}$ of L^2 inputs that make the output L^2. Another common choice of the domain of optimization is $\mathcal{U}_{\mathrm{exp}}$, the set of L^2 inputs that make the state L^2. (By [WR01], $\mathcal{U}_{\mathrm{exp}}(x_0) \subset \mathcal{U}(x_0)$ for every $x_0 \in \mathtt{X}$.) That means that one wants to minimize the cost over this smaller set of inputs, so that the minimum may become strictly bigger. In the finite-dimensional setting, $\mathcal{U}_{\mathrm{exp}}$ is almost exclusively used. However, our methods apply also to that setting and even to more general settings, such as the set of strongly stabilizing controls, studied in, e.g., [Oos00]. This generalized framework was presented in [Mik02] and will be extended in [Mik05a], covering also indefinite problems.

In Lemma 4.1 we showed that certain admissible (or well-posed) state feedback minimizes the cost (1.10) over $\mathcal{U}$. In [Mik05b] we shall show that this applies to any cost function of form (3.2) as long as the generalized Popov Toeplitz operator is uniformly positive (or equivalently, $\mathcal{J}(0,u) \ge \epsilon(\|y\|_2^2+\|u\|_2^2)$) and also give the analogous result for minimization over $\mathcal{U}_{\mathrm{exp}}$. The sufficiency is well known for stable systems [WW97] [Sta97] (for the $\mathcal{U}$ case), where an equivalent condition is that the *Popov function* $\hat{\mathscr{D}}^*J\hat{\mathscr{D}}$ is uniformly positive (or $\hat{\mathscr{D}}^*J\hat{\mathscr{D}} \ge \epsilon I$ a.e. on the imaginary axis in the separable case).

To some extent the properties of and differences between coprimeness and quasi-coprimeness were explored in [Mik02]. We shall present further details and results in later articles. We shall also show that quasi-coprime factorization can be applied to establish several new results on dynamic stabilization.

Appendix A. Miscellaneous results

The following is a well-known, simple consequence of the Closed-Graph Theorem:

Lemma A.1. *Assume that X_1, X_2 and X_3 are Banach spaces and $X_2 \subset X_3$ continuously.*

If $T \in \mathcal{B}(X_1, X_3)$ and $T[X_1] \subset X_2$, then $T \in \mathcal{B}(X_1, X_2)$. □

By L^2_c we denote the functions $u \in \mathrm{L}^2$ with compact support. Recall from Definition 2.1 that $\mathscr{C}$ is stable iff $\mathscr{C}[\mathtt{X}] \subset \mathrm{L}^2$ (by Lemma A.1). It follows that $\mathscr{D}$ is "almost stable":

Lemma A.2. *If $\mathscr{C}$ is stable, then $\mathscr{D}[\mathrm{L}^2_\mathrm{c}] \subset \mathrm{L}^2$.*

Proof. Let $u \in \mathrm{L}^2_\mathrm{c}(\mathbb{R}; U)$ and $T > 0$ be such that $\pi_+\tau^T u = 0$. Then $\mathscr{D}u = \tau^{-T}\mathscr{D}\tau^T u = \tau^{-T}\mathscr{C}\mathscr{B}\tau^T u \in \mathrm{L}^2$. □

It is a well-known consequence of the Liouville Theorem that if a (causal) map $\mathscr{E} \in \mathrm{TIC}_0(\mathtt{U}, \mathtt{Y})$ is also "anti-causal" ($\pi_+\mathscr{E}\pi_- = 0$), then $\mathscr{E} \in \mathcal{B}(\mathtt{U})$, i.e., then there exists $E \in \mathcal{B}(\mathtt{U})$ such that $(\mathscr{E}u)(t) = Eu(t)$ a.e. for every $u \in \mathrm{L}^2(\mathbb{R}; \mathtt{U})$. We need the following generalization of this result:

Lemma A.3. *If $\mathscr{M} \in \mathcal{G}\mathrm{TIC}_\infty$, $\mathscr{E} \in \mathrm{TIC}_\infty$ and $\pi_+\mathscr{M}\pi_+\mathscr{E}\pi_- = 0$, then $\mathscr{E} \in \mathcal{B}$.*

Proof. 1° *Case $\mathscr{M} = I$:* Case $\mathscr{E} \in \mathrm{TIC}_0$ is [Sta97, Lemma 6]. For general $\mathscr{E} \in \mathrm{TIC}_\omega$, we can replace $\mathscr{E}$ by $\mathrm{e}^{-\omega\cdot}\mathscr{E}\mathrm{e}^{\omega\cdot} \in \mathrm{TIC}_0$.

2° *General case:* Now $\pi_+\mathscr{E}\pi_- = \pi_+\mathscr{M}^{-1}(\pi_+)\mathscr{M}\pi_+\mathscr{E}\pi_- = 0$ (use the fact that $\pi_+\mathscr{M}\pi_+ = \mathscr{M}\pi_+$, because $\pi_+\mathscr{M}\pi_- = 0$), hence $\mathscr{E} \in \mathcal{B}$, by 1°. □

Consequently, when $\mathscr{E} \in \mathrm{TIC}_0$ and $\mathscr{E} = \mathscr{E}^*$, we have $\mathscr{E} \in \mathcal{B}$ (because $\pi_+\mathscr{E}\pi_- = (\pi_-\mathscr{E}^*\pi_+)^* = 0$). However, we also need a similar result for unstable maps (of form "$\mathscr{D}^*J\mathscr{D}$") in place of $\mathscr{E}$:

Lemma A.4 ("$\mathscr{D}^*J\mathscr{D} = S$"). *Let $\mathscr{D} \in \mathrm{TIC}_\infty(\mathtt{U}, \mathtt{Y})$ and $J = J^* \in \mathcal{B}(\mathtt{Y})$. Assume that $\mathscr{D}u \in \mathrm{L}^2$ and $\langle \mathscr{D}\pi_+v, J\mathscr{D}\pi_-u\rangle = 0$ for all $u, v \in \mathrm{L}^2_\mathrm{c}$. Then there exists a unique $S = S^* \in \mathcal{B}(\mathtt{U})$ such that $\langle \mathscr{D}v, J\mathscr{D}u\rangle = \langle v, Su\rangle$ for all $u, v \in \mathrm{L}^2_\mathrm{c}$.* □

In the proof below we show that the operators $\mathscr{S}_t := (\mathscr{D}\pi_{[-t,t)})^*J\mathscr{D}\pi_{[-t,t)} \in \mathcal{B}(\mathrm{L}^2([-t,t); \mathtt{U}))$ are restrictions of $\mathscr{S}_T$ ($T \ge t \ge 0$) and can be extended to a static operator S.

Proof. In the sequel we shall use the fact that if $u \in \mathrm{L}^2_\mathrm{loc}$ and $\langle v, u\rangle = 0$ for all $v \in \mathrm{L}^2_\mathrm{c}$ then $u = 0$ (a.e.). This fact also implies that S is unique.

Let $s, t \in \mathbb{R}$. Replace u by $\tau^t u$ to obtain that $\langle \mathscr{D}\pi_{[t,\infty)}v, J\mathscr{D}\pi_{(-\infty,t)}u\rangle = 0$ for all $u, v \in \mathrm{L}^2_\mathrm{c}$ (use the facts that $\pi_-\tau^t = \tau^t\pi_{(-\infty,t)}$, $(\tau^t)^* = \tau^{-t}$ and $\tau^{-t}\pi_+ = \pi_{[t,\infty)}\tau^{-t}$). Because $J = J^*$, we have $\langle \mathscr{D}\pi_{(-\infty,s)}v, J\mathscr{D}\pi_{[s,\infty)}u\rangle = 0$ for all $u, v \in \mathrm{L}^2_\mathrm{c}$, hence

$$\langle \mathscr{D}v, J\mathscr{D}\pi_{[s,t)}u\rangle = \langle \mathscr{D}\pi_{[s,t)}v, J\mathscr{D}\pi_{[s,t)}u\rangle \quad (u, v \in \mathrm{L}^2_\mathrm{c},\ -\infty \le s \le t \le +\infty). \quad \text{(A.1)}$$

We have $\mathscr{D}\pi_{[-t,t)} \in \mathcal{B}(\mathrm{L}^2([-t,t); \mathtt{U}), \mathrm{L}^2(\mathbb{R}; \mathtt{U}))$, by the assumption and Lemma A.1. Set

$$\mathscr{S}_t := (\mathscr{D}\pi_{[-t,t)})^*J\mathscr{D}\pi_{[-t,t)} \in \mathcal{B}(\mathrm{L}^2([-t,t); \mathtt{U})) \quad (t > 0). \quad \text{(A.2)}$$

Then $\langle v, \mathscr{S}_t u\rangle = \langle \mathscr{D}v, J\mathscr{D}u\rangle$ for $u, v \in \pi_{[-t,t)}\mathrm{L}^2$, hence for $u \in \pi_{[-t,t)}\mathrm{L}^2$ and $v \in \mathrm{L}^2_\mathrm{c}$, by (A.1). Consequently, for any $u \in \pi_{[-t,t)}\mathrm{L}^2$ and any $T > t > 0$, we have $\mathscr{S}_T u = \mathscr{S}_t u \in \pi_{[-t,t)}\mathrm{L}^2$. Therefore, we can define $\mathscr{S}u := \mathscr{S}_t u$ for any $t > 0$ and $u \in \mathrm{L}^2([-t,t); \mathtt{U})$.

It follows that $\mathscr{S} : \mathrm{L}^2_\mathrm{c} \to \mathrm{L}^2_\mathrm{c}$, $\mathscr{S}_T = \mathscr{S}\pi_{[-T,T)} = \pi_{[-T,T)}\mathscr{S}$, and $\tau\mathscr{S} = \mathscr{S}\tau$ (by (A.1)). Therefore,

$$\|\mathscr{S}u\|_2^2 = \sum_{n\in 2\mathbb{Z}} \|\pi_{[n-1,n+1)}\mathscr{S}u\|_2^2 = \sum_{n\in 2\mathbb{Z}} \|\tau^{-n}\pi_{[-1,1)}\tau^n\mathscr{S}u\|_2^2 \tag{A.3}$$

$$= \sum_{n\in 2\mathbb{Z}} \|\tau^{-n}\mathscr{S}\pi_{[-1,1)}\tau^n u\|_2^2 = \sum_{n\in 2\mathbb{Z}} \|\mathscr{S}\pi_{[-1,1)}\tau^n u\|_2^2 \tag{A.4}$$

$$\leq \|\mathscr{S}_1\|_{\mathcal{B}(\mathrm{L}^2)}\| \sum_{n\in 2\mathbb{Z}} \|\pi_{[-1,1)}\tau^n u\|_2^2 = \|\mathscr{S}_1\|_{\mathcal{B}(\mathrm{L}^2)}\|u\|_2^2 \tag{A.5}$$

for $u \in \mathrm{L}^2_\mathrm{c}$. Consequently, $\mathscr{S}$ can be extended to a $\mathcal{B}(\mathrm{L}^2)$ map that satisfies $\mathscr{S}\tau = \tau\mathscr{S}$. From (A.1) it follows that $\pi_+\mathscr{S}\pi_- = 0 = \pi_-\mathscr{S}\pi_+$, hence $\mathscr{S} = S \in \mathcal{B}(\mathtt{U})$, by [Sta97, Lemma 6]. Obviously, $\mathscr{S} = \mathscr{S}^*$, hence $S = S^*$. □

The (closed-loop) state-to-control map $\mathscr{F}_\circlearrowleft$ (see (2.6)) determines the pair $[\ \mathscr{F} \mid \mathscr{G}\ ]$ uniquely modulo a unit constant:

Lemma A.5 (All $[\ \mathscr{F} \mid \mathscr{G}\ ]$). *Let $[\ \mathscr{F} \mid \mathscr{G}\ ]$ be an admissible state-feedback pair for Σ. Then all admissible state-feedback pairs $[\ \tilde{\mathscr{F}} \mid \tilde{\mathscr{G}}\ ]$ leading to same $\mathscr{F}_\circlearrowleft$ are given by*

$$[\ \tilde{\mathscr{F}} \mid \tilde{\mathscr{G}}\ ] = [\ Q\mathscr{F} \mid I - Q(I-\mathscr{G})\ ] \qquad (Q \in \mathcal{GB}(\mathtt{U})). \tag{A.6}$$

When F is bounded, the above claim is rather obvious (see [Mik02, p. 800]). Therefore, discretization (see [Mik02]) leads to an alternative proof of the lemma.

We observe that the whole left column of the closed-loop system (2.6) is the same for all such pairs; the only difference on the right column corresponds to the coordinate change $u_\circlearrowleft \mapsto Qu_\circlearrowleft$ in the exogenous input.

Proof. Define $\tilde{\mathscr{M}} := (I - \tilde{\mathscr{G}})^{-1}$ and $\tilde{\Sigma}_\circlearrowleft$ for $[\ \tilde{\mathscr{F}} \mid \tilde{\mathscr{G}}\ ]$ as in Definition 2.4. Set $\mathscr{E} := \mathscr{M}^{-1}\tilde{\mathscr{M}} \in \mathcal{G}\mathrm{TIC}_\infty(\mathtt{U})$. By (2.6), (2.5) and (twice) 4. of Definition 2.1, we have

$$\pi_+\mathscr{M}\mathscr{E}\pi_- = \pi_+\tilde{\mathscr{M}}\pi_- = \tilde{\mathscr{F}}_\circlearrowleft\tilde{\mathscr{B}}_\circlearrowleft = \mathscr{F}_\circlearrowleft\tilde{\mathscr{B}}_\circlearrowleft = \mathscr{M}\mathscr{F}\tilde{\mathscr{B}}\tilde{\mathscr{M}} = \mathscr{M}\pi_+\mathscr{G}\pi_-\tilde{\mathscr{M}} \tag{A.7}$$

$$= (\pi_+)\mathscr{M}\pi_+(I - \mathscr{M}^{-1})\pi_-\tilde{\mathscr{M}} = -\pi_+\mathscr{M}(I-\pi_-)\mathscr{M}^{-1}\pi_-\tilde{\mathscr{M}} \tag{A.8}$$

$$= -0 + \pi_+\mathscr{M}\pi_-\mathscr{M}^{-1}(\pi_-)\tilde{\mathscr{M}} = \pi_+\mathscr{M}\pi_-\mathscr{E}(\pi_-). \tag{A.9}$$

Therefore, $\pi_+\mathscr{M}(I-\pi_-)\mathscr{E}\pi_- = 0$, hence $E := \mathscr{E} \in \mathcal{B}(\mathtt{U})$, by Lemma A.3. Since $\mathscr{E} \in \mathcal{G}\mathrm{TIC}_\infty(\mathtt{U})$, the operator E is invertible and onto, hence $E \in \mathcal{GB}(\mathtt{U})$.

Since $\tilde{\mathscr{M}} = \mathscr{M}E$, set $Q := E^{-1}$ to have $\tilde{\mathscr{M}}^{-1} = Q\mathscr{M}^{-1}$, $\tilde{\mathscr{G}} = I - \tilde{\mathscr{M}}^{-1} = I - Q(I-\mathscr{G})$ and $\tilde{\mathscr{F}} = \tilde{\mathscr{M}}^{-1}\mathscr{F}_\circlearrowleft = Q\mathscr{F}$. □

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