REAL SOLUTIONS TO CONTROL, APPROXIMATION AND FACTORIZATION PROBLEMS[*]

KALLE M. MIKKOLA[†]

AMS subject classifications. 93D25, 49N10, 93D15, 47B35, 47A68

Key words. Real-symmetric functions, real coprime factors, real state-feedback operators, real stabilizing controllers, real optimal control, real Hilbert spaces

Abstract. During the past decades much of finite-dimensional systems theory has been generalized to infinite dimensions. However, there is one important flaw in this theory: it only guarantees complex solutions, even when the data is real. We show that the standard solutions of many classical problems with real data are also real.

We call a (possibly matrix- or operator-valued) holomorphic function G *real* (real-symmetric) if $G(\bar{z}) = \overline{G(z)}$ for every z. We show that if such a function can be presented as $G = NM^{-1}$, where $N, M \in \mathcal{H}^\infty$, then we have $G = N_R M_R^{-1}$, where $N_R, M_R \in \mathcal{H}^\infty$ are real and weakly right coprime.

Consequently, if a real function G has a *stabilizing compensator* (i.e., a function K such that $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in \mathcal{H}^\infty$), then G has a real doubly coprime factorization and a Youla parameterization of all real stabilizing controllers.

If a *system* of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$ or of the form $x_{n+1} = Ax_n + Bu_n$, $y_n = Cx_n + Du_n$ has real (possibly unbounded, constant) coefficients A, B, C and D, then the system is stabilizable iff it is stabilizable by a real state-feedback operator. This holds for both exponential stabilization and output stabilization. A real stabilizing state-feedback operator is then given by the standard LQR feedback operator, hence the standard (complex) formulae can be used to find this real solution. Analogous results are established for other optimization, factorization, approximation and representation problems too.

1. Introduction.

During the past decades much of finite-dimensional systems theory has been generalized to infinite dimensions. However, there is one important flaw in this theory: it usually only guarantees complex solutions, even when the data is real. For applications, complex solutions are impossible to implement; one needs solutions that are real numbers, real sequences, real-symmetric functions—or that are matrices (or operators) having such entries.

Consequently, it is essential to develop a theory that guarantees real solutions from real data. The aim of this paper is to show that this is possible for a wide range of classical control problems. We show how for many output-feedback, state-feedback and other control problems, standard methods yield real solutions if the original system or transfer function is real (that is, real-symmetric: $G(\bar{\cdot}) = \overline{G(\cdot)}$). Both state-space and frequency-domain problems are treated, including optimal control, stabilization, factorization, approximation and representation.

We cover weakly coprime and Bézout coprime factorizations, Youla parameterization of stabilizing compensators (for dynamic output feedback), exponential stabilization and output-stabilization by state feedback, the LQR problem and other, possibly indefinite optimal control problems (such as the $\mathcal{H}^\infty$ minimax control), spectral factorization, the Nehari Theorem etc.

In Section 2 we give the exact definition of "*real*". Then we show that if a real function has a *weakly coprime factorization*, then it has a weakly coprime factorization with real factors. If it has a coprime factorization, then it has a real doubly coprime

factorization and the corresponding Youla formula parameterizes all real stabilizing controllers, that is, all real functions K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$. We recall that also the converse holds [Ino88] [Smi89] [Mik07a]: if a function has a stabilizing controller, then it has a coprime factorization.

A related problem, namely the existence of "stable" (that is, $K \in \mathcal{H}^\infty$) real stabilizing compensators, have been studied in, e.g., [MW09], [Wic10] and [Sta92], and Bass stable rank for real-$\mathcal{H}^\infty$ is 2 [MW09].

The real versions of Tolokonnikov's Lemma and of the inner-outer factorization were established in [MS07]. For the Corona Theorem, the symmetrization of any solution yields a solution (i.e., a left inverse). In Section 3 we show that the same symmetrization method applies to the Hartman and Nehari Theorems and that other methods yield real spectral factorization. Also further results on real-symmetric functions are obtained for later use.

Discrete-time systems and state feedback are defined in Section 4: the "next state" equation is $x_{n+1} = Ax_n + Bu_n$ with the initial state x_0 given, u being the input sequence, and $y_n = Cx_n + Du_n$ being the output of the system.

In Section 5 we show that if a real system is *output-stabilizable* by state feedback, then the "LQ-optimal" state-feedback operator is real. This provides a real output-stabilizing state-feedback operator for the system. Moreover, if a real system is power stabilizable, then it is power stabilizable by a real state-feedback operator. On the other hand, the LQ-optimal control always determines a "canonical" weakly coprime factorization of the transfer function; this canonical factorization is then real too. Corresponding proofs are given in Section 6, where analogous "real results" are given also for *indefinite cost functions*.

In Section 7 we show that every real holomorphic function defined on a neighborhood of the origin has a real realization. Using this and the results of Section 5 we prove the results of Section 2.

Above we refer to discrete-time systems, but essentially all results of previous sections hold for *continuous-time systems* too (where $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(0)$ given, and A and B possibly unbounded), as shown in Section 8.

In the accompanying report [Mik10], further details are given, Banach-space-valued functions are treated, and "real variants" are established also for the standard Hankel and Toeplitz operator results of [Mik09a] and [Mik07b], including the Lax–Halmos Theorem and the $\mathcal{H}^2_{\text{strong}}$ inner–outer factorization.

Notation. The following notation is defined later in the following order (the word "real" thrice for different objects).
Section 2: $\mathcal{B}(\mathtt{X},\mathtt{Y})$, $\mathcal{H}^\infty$; $\mathbb{D}$, $\mathbb{T}$, $\mathbb{N}$; $\mathtt{U}$, $\mathtt{X}$, $\mathtt{Y}$; $\mathtt{U}_\mathbb{R}$, "*real*"; ℓ^2; Re, Im, $i = \sqrt{-1}$, $\overline{u}$; "*real*", $\mathcal{B}_\mathbb{R}$, A_R, A_I, $\overline{A}$; $\overline{\Omega}$; "*real*", "*real-symmetric*", $\mathcal{H}^\infty_\mathbb{R}$; "*proper*", "*right coprime*", "*weakly right coprime*", "*normalized*".
Section 3: f_R, f_I; $\mathrm{L}^\infty_{\text{strong}}$;
Section 4: "*system*" $(\frac{A\,|\,B}{C\,|\,D})$, "*transfer function*" G, "*realization*"; Z-transform $\widehat{u}$; "*state-feedback*" F; "*closed-loop system*", N, M; "*output-stable*", "*power-stable*".
Section 5: "*LQR, LQ*", "*Finite Cost Condition*".
Section 6: J, "*cost function* $\mathcal{J}(x_0, u)$", "*J-minimal*"; $\mathscr{C}, \mathscr{D}$; $\mathcal{U}(x_0)$; "*J-optimal*"; $\mathcal{U}_\mathbb{R}(x_0)$, $\mathrm{Re}\,u$; "*J-optimal cost operator*" $\mathcal{P}$; "*J-optimal state-feedback*".
Section 8: $\mathbb{C}^+$ and continuous-time terminology.

2. Coprime factorization and stabilizing compensators. It is known that all fractions NM^{-1}, $N, M \in \mathcal{H}^\infty$ can be reduced so that N and M are "weakly coprime" (no common factors except units). They can be made "strongly coprime"

(Bézout coprime) iff the function $G := NM^{-1}$ has a "stabilizing compensator", i.e., a function K such that $\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$. We present the corresponding definitions and details in this section, and, as the new result, we show that one can always find real factors and compensators. We start with definitions of "real" etc.

By $\mathcal{B}(\mathtt{X},\mathtt{Y})$ we denote the Banach space of bounded linear operators $\mathtt{X} \to \mathtt{Y}$; by $\mathcal{H}^\infty(\mathtt{Z})$ we denote the Banach space of bounded holomorphic functions $\mathbb{D} \to \mathtt{Z}$, where $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ is the unit disc, $\mathtt{X}$ and $\mathtt{Y}$ are Hilbert spaces and $\mathtt{Z}$ is a Banach space. We set $\mathcal{B}(\mathtt{X}) := \mathcal{B}(\mathtt{X},\mathtt{X})$, $\mathbb{T} := \{z \in \mathbb{C} \,\big|\, |z| = 1\}$, $\mathbb{N} := \{0,1,2,\dots\}$.

In this article $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ denote complex Hilbert spaces with fixed real-linear subspaces $\mathtt{U}_\mathbb{R}$, $\mathtt{X}_\mathbb{R}$ and $\mathtt{Y}_\mathbb{R}$ such that $\mathtt{U} = \mathtt{U}_\mathbb{R} + i\mathtt{U}_\mathbb{R}$, $\mathtt{U}_\mathbb{R} \cap i\mathtt{U}_\mathbb{R} = \{0\}$, and $\langle u,v \rangle \in \mathbb{R}$ for every $u,v \in \mathtt{U}_\mathbb{R}$; similarly for $\mathtt{X}$ and $\mathtt{Y}$. Obviously, $\mathtt{U}_\mathbb{R}, \mathtt{X}_\mathbb{R}, \mathtt{Y}_\mathbb{R}$ are then real Hilbert spaces.

We call the elements of $\mathtt{U}_\mathbb{R}$, $\mathtt{X}_\mathbb{R}$ and $\mathtt{Y}_\mathbb{R}$ *real*. For $\mathbb{C}^n$, ℓ^2, etc. we use standard definitions; e.g., $(\mathbb{C}^n)_\mathbb{R} = \mathbb{R}^n$ and $\ell^2(\mathbb{N};\mathbb{C})_\mathbb{R} = \ell^2(\mathbb{N};\mathbb{R})$, so by real elements of ℓ^2 we mean real-valued sequences. However, to make it simple, the reader could consider our "input/output" dimensions finite (i.e., $\mathtt{U} = \mathbb{C}^n$, $\mathtt{Y} = \mathbb{C}^m$, with $\mathtt{U}_\mathbb{R} = \mathbb{R}^n$, $\mathtt{Y}_\mathbb{R} = \mathbb{R}^m$), as the main results seem to be new even in that setting.

The projections $\operatorname{Re}, \operatorname{Im} : \mathtt{U} \to \mathtt{U}_\mathbb{R}$ defined by $u = \operatorname{Re} u + i \operatorname{Im} u$ are unique, so also the *conjugate* $\overline{u} := \operatorname{Re} u - i \operatorname{Im} u$ is well defined. We obviously have $\|u + iv\|^2 = \|u\|^2 + \|v\|^2$ for real $u,v \in \mathtt{U}$.

An operator $A \in \mathcal{B}(\mathtt{U},\mathtt{Y})$ is called *real* ($A \in \mathcal{B}_\mathbb{R}$) if $Au \in \mathtt{Y}_\mathbb{R}$ for each $u \in \mathtt{U}_\mathbb{R}$. If $\mathtt{U} = \mathbb{C}^n$, $\mathtt{Y} = \mathbb{C}^m$, then this obviously holds iff A is a real matrix. Also in the general case, $\mathcal{B}_\mathbb{R}$ is a real Banach space, and the equation $A = A_R + iA_I$ defines unique projections $\mathcal{B} \to \mathcal{B}_\mathbb{R}$, so we can define $\overline{A} := A_R - iA_I$. One can show that $A_R u = \operatorname{Re}(A \operatorname{Re} u) + i \operatorname{Re}(A \operatorname{Im} u)$ ($u \in \mathtt{U}$). In particular, $A = A_R$ iff A is real.

One easily verifies that $\overline{\alpha A + B} = \overline{\alpha}\overline{A} + \overline{B}$, $\overline{AB} = \overline{A}\,\overline{B}$, $\overline{(\overline{A})} = A$, $\overline{A}^* = \overline{A^*}$, $\overline{A^{-1}} = (\overline{A})^{-1}$, $\|\overline{A}\| = \|A\|$, when $\alpha \in \mathbb{C}$ and A and B are linear operators or vectors of compatible dimensions.

Example. The matrix $A = \left[\begin{smallmatrix} 2 & i \\ -i & 2 \end{smallmatrix}\right] = A^* \geq I$ is positive but not real: $A_R = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, $A_I = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

A basic reference on real operator algebras is [Li03].

For any $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function f we have $f \in \mathcal{H}^\infty$ iff $\overline{f(\overline{\cdot})} \in \mathcal{H}^\infty$. If $\Omega \subset \mathbb{C}$ is a set, then $\overline{\Omega} := \{\overline{z} \,\big|\, z \in \Omega\}$ denotes the set of complex conjugates of the elements of Ω.

A function $\sum_{k=0}^\infty a_k z^k$ is called real iff the coefficients a_k are real. An equivalent definition is given below.

DEFINITION 2.1 (real). *Let $\Omega = \overline{\Omega} \subset \mathbb{C}$ be open. A holomorphic function $f : \Omega \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ is called* real *(or real-symmetric) if*

$$f(z) = \overline{f(\overline{z})} \qquad (z \in \Omega). \tag{2.1}$$

By $\mathcal{H}_\mathbb{R}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$ we denote the set (the real Banach space) of real elements of $\mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$.

A vector, matrix or sequence is called real *if its elements are real.*

An element $f \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U},\mathtt{Y}))$ is real iff its Fourier coefficients are real (by Lemma 3.1 below), or equivalently, iff it is the Z-transform of a real sequence $\mathbb{N} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$. One more equivalent condition is that $f(z)$ is real for real z.

One observes that a constant (possibly operator-valued) function is real-symmetric if and only if its value is real.

If we fix orthonormal bases of $\mathtt{U}_\mathbb{R}$ and $\mathtt{Y}_\mathbb{R}$ (such are necessarily also orthonormal bases of $\mathtt{U}$ and $\mathtt{Y}$), then the function f in (2.1) can be written as a matrix (f_{ij}), where

the indices i, j run over the (possibly uncountable) bases. Obviously, the function f is real-symmetric iff every f_{ij} is real-symmetric.

Next we define coprime factorizations. Recall that with the set of integers in place of $\mathcal{H}^\infty$, both (b) and (c) below are well-known properties of integers. Moreover, (b) and (c) are equivalent for rational functions but not for general $\mathcal{H}^\infty$ functions [Smi89].

DEFINITION 2.2 (Coprime). *Let $N \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$ and $M \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$.*

(a) *A function defined and holomorphic on a neighborhood of 0 is called* proper.

(b) *We call N and M* right coprime *if $AM - BN \equiv I$ on $\mathbb{D}$ for some $A, B \in \mathcal{H}^\infty$.*

(c) *We call N and M* weakly right coprime[1] *[Smi89] [Mik09b] if $Mf, Nf \in \mathcal{H}^\infty \implies f \in \mathcal{H}^\infty$ for every proper holomorphic $\mathtt{U}$-valued function f.*

(d) *We call N and M* normalized *if $\begin{bmatrix} N \\ M \end{bmatrix}$ is inner (i.e., if $\| \begin{bmatrix} N \\ M \end{bmatrix} u_0 \| = \| u_0 \|$ a.e. on $\mathbb{T}$ for every $u_0 \in \mathtt{U}$).*

Any quotient N/M of integers N, M can be reduced so that N and M are relative primes ($\gcd(N, M) = 1$). Similarly, any real function NM^{-1} can be written so with N and M real, weakly right coprime and normalized:

THEOREM 2.1. *Let $N \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$, $M \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}))$, and let $M(0)$ be invertible.*

If the function NM^{-1} is real, then there exist $N_c \in \mathcal{H}^\infty_\mathbb{R}(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$, $M_c \in \mathcal{H}^\infty_\mathbb{R}(\mathcal{B}(\mathtt{U}))$ such that $M_c(0)$ is invertible, $NM^{-1} = N_c M_c^{-1}$ on a neighborhood of 0, and N_c and M_c are normalized and weakly right coprime.

If N and M are right coprime, then so are N_c and M_c.

As shown in the proof (Theorems 2.1, 2.2, and 2.3 are proved in Section 7), we can use the standard LQR constructive formulae for N_c and M_c, thus using the "LQR Riccati equation".

If $\dim \mathtt{U} < \infty$, then the N and M in Theorem 2.1 are weakly right coprime iff $\gcd(N, M) = 1$ [Smi89] [Mik08, Theorem 2.16], i.e., iff all common divisors are units, that is, $M = AX$, $N = BX$, $A, B \in \mathcal{H}^\infty$, $X \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U})) \implies X^{-1} \in \mathcal{H}^\infty$. Further equivalent characterizations of weak coprimeness are given in [Mik09b] and [Mik08]. Naturally, we may replace 0 by any $\alpha \in \mathbb{D}$ in Theorem 2.1.

Any stabilizable real transfer function is stabilizable by a real compensator:

THEOREM 2.2 (Stabilizing compensator). *Let G be a real proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function. If there exists a proper $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued function K such that $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$, then there exists a real proper $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued function K such that $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \in \mathcal{H}^\infty_\mathbb{R}(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$.*

Further details on (internal, or dynamic output-feedback) stabilization are given in, e.g., [Mik07a], [Smi89] and [Vid85].

Using the above results, we can present the Youla parameterization of all real stabilizing compensators for G.

THEOREM 2.3 (Youla parameterization). *Let G be a real proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function. The condition in Theorem 2.2 holds iff $G = NM^{-1}$, where $M(0)$ is invertible in $\mathcal{B}(\mathtt{U})$ and N and M are right coprime. If the condition holds, then N and M can be chosen so that they are real, by Theorem 2.1. Assume that such real N and M exist.*

Then there exist real $X, Y \in \mathcal{H}^\infty$ such that $X(0)$ is invertible and $\begin{bmatrix} M & Y \\ N & X \end{bmatrix}$ is invertible in $\mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$. Moreover, all proper $\mathcal{B}(\mathtt{Y}, \mathtt{U})$-valued functions K satisfying

---

[1]Equivalence with the standard definition requires coercivity at 0. This difference is redundant in this article, because in applications we have $M(0)$ invertible. Moreover, in the operator-valued case this definition is more useful. Note: when f is a holomorphic function $\Omega \to \mathtt{U}$, we mean by "$f \in \mathcal{H}^\infty$" that $f|_{\Omega \cap \mathbb{D}}$ is the restriction of an element of $\mathcal{H}^\infty(\mathtt{U})$.

$\left[\begin{smallmatrix} I & -K \\ -G & I \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$ *are given by the* Youla parameterization

$$K = (Y + MQ)(X + NQ)^{-1} \tag{2.2}$$

where $Q \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{Y}, \mathtt{U}))$ *is such that* $(X + NQ)^{-1}$ *is proper. The map* $Q \mapsto K$ *in* (2.2) *is one-to-one. The function* K *is real iff* Q *is real.*

In some engineering applications one might wish to use (real) non-proper controllers [CWW01] [WC97], which are parameterized by (2.2) without the requirement that $(X + NQ)^{-1}$ is proper [Mik07a, Theorem 1.1 and Section 3]. The following remark parameterizes all real controllers in this generalized sense.

REMARK 2.3. Theorem 2.3 holds even if we remove "$X(0)$ is invertible and", as one observes from the proof. Thus, any real extension of $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ to an invertible element of $\mathcal{H}^\infty$ will do in the theorem. ◁

In the matrix-valued case (dim $\mathtt{U}$, dim $\mathtt{Y} < \infty$) it is always possible to have $K \in \mathcal{H}^\infty$ ("stabilization by a stable controller"), but we cannot require that K to be real unless the real poles and zeros of G satisfy the "positive on real zeros" condition (or "parity interlacing condition"), in which case the problem was solved in [Wic10] in the scalar-valued case. Unlike in that problem, in the problems studied in this article the existence of a solution always implies the existence of a real solution.

The domains of M^{-1} and G require some attention in the operator-valued case:

REMARK 2.4 (domains of M^{-1} and G). If dim $\mathtt{U} < \infty$ and $M(0)$ is invertible, then $\det M$ and hence also M is invertible on $\mathbb{D}$ minus some isolated points. If dim $\mathtt{U} = \infty$, then one has to be particularly careful with the (possibly disconnected) domains of M^{-1}, G and K in Theorem 2.3. One way to solve this problem would be to consider "=" and "∈" on sufficiently small neighborhoods of the origin only. However, if G and K are holomorphic on any open and connected $\Omega \subset \mathbb{D}$, then the equations $G = NM^{-1}$ and (2.2) actually hold on Ω. In particular, then M and $X + NQ$ are invertible on Ω. [Mik07a, Lemma 6.1] ◁

There are several explicit formulae for N, M, X and Y in the literature, mostly corresponding to the solutions of Riccati equations corresponding to an arbitrary output-stabilizable realization of G. We refer below to the most general formulae and observe that their yields become real if G is real.

REMARK 2.5 (Constructive formulae). Explicit formulae for N, M, X and Y and robust stabilizing compensators are provided in, e.g., [CO06] and [Cur06] for continuous time and in [CO11] for discrete time.

All these formulae are given in terms of a realization Σ of G such that Σ and its dual are output-stabilizable. A constructive algorithm for finding such a realization is given in [Mik09b, Remark 5.3]. Moreover, that algorithm and the formulae mentioned above yield real results if the data is real, by Theorems 7.1 and 5.2, which themselves yield an algorithm for real coprime factorizations. ◁

3. Real operators. In this section we further elaborate the concept "real" and obtain related results used in the later sections. We also show the existence of real solutions to the Nehari, Hartman, and spectral factorization problems (provided that the data is real and a complex solution exists).

We first recall some equivalent characterizations of real-symmetric functions from (the proof of) [MS07, Lemma 2.1]. The reader can take here, e.g., $\Omega = \mathbb{D}$.

LEMMA 3.1. *Let* $f : \Omega \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ *be holomorphic and* $\Omega = \overline{\Omega} \subset \mathbb{C}$ *open and connected. If* $\Omega \cap \mathbb{R} \neq 0$, *then* f *is real (i.e.,* $f = \overline{f(\bar{\cdot})}$*) iff* $f(z)$ *is real for each* $z \in \Omega \cap \mathbb{R}$ *(or on a nondiscrete subset of* $\Omega \cap \mathbb{R}$*). If* $0 \in \Omega$, *then* f *is real if and only if every Taylor series (at* 0*) coefficient* $\widehat{f}(n)$ *is real (*$n \in \mathbb{N}$*).*

Next we record a few more facts on real elements. Here any functions have a Lebesgue-measurable domain $Q \subset \mathbb{C}$ such that $Q = \overline{Q}$ (and the dimensions are assumed to be compatible in (d)).

LEMMA 3.2. **(a)** *The functions (or constants)* $f_R := \frac{1}{2}(f + \overline{f(\bar{\cdot})})$ *and* $f_I := (-if)_R$ *are real and* $f = f_R + if_I$ *when* f *is a function (or constant) with values in* $\mathbb{C}$, $\mathtt{U}$ *or* $\mathcal{B}(\mathtt{U}, \mathtt{Y})$. *Moreover,* f_R *and* f_I *are unique,* $\overline{f(\bar{\cdot})} = f_R - if_I$, *and* $\overline{f(z)} = f_R(\bar{z}) - if_I(\bar{z})$.

(b) *If* $f \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$, *then* $f_R(z) = \sum_{n=0}^\infty \widehat{f}(n)_R z^n$ *and* $f_I(z) = \sum_{n=0}^\infty \widehat{f}(n)_I z^n$.

(c) *If* $f \in \mathrm{L}^1(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ *is real-symmetric, then the Fourier coefficients* $\widehat{f}(n) :=$ $\frac{1}{2\pi} \int_{-\pi}^\pi \mathrm{e}^{-in\theta} f(\mathrm{e}^{i\theta}) \, d\theta$ *are real operators.*

(d) *If* $g = g_R$, *then* $(fg)_R = f_R g$ *and* $(gh)_R = g h_R$.

(e) *If* $u, v \in \mathtt{U}$ *are real, then* $\|u + iv\|^2 = \|u\|^2 + \|v\|^2$.

(f) *If* $A \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$ *is real, then* $\|A\| = \sup_{u = u_R \in \mathtt{U}, \|u\| \le 1} \|Au\|_\mathtt{Y}$.

(g) *If* g *is a real one-to-one map of* Ω_1 *onto* Ω_2, *where* $\Omega_1, \Omega_2 \subset \mathbb{C}$, $\Omega_1 = \overline{\Omega_1}$, *then* g^{-1} *is real. Moreover, then a function* $h : \Omega_2 \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ *is real iff* $h \circ g$ *is real.*

(The proofs of Lemmata 3.2 and 3.3 are given in Appendix A.)

From (b) we deduce that if f is holomorphic (or bounded) on $\mathbb{D}$ or on a right half-plane, then so are $\overline{f(\bar{\cdot})}$, f_R and f_I. However, for $f(z) = 1 + iz$ we have $f_R(i) = 1$ but $f(i) = 0$, so f_R is not pointwise bounded by f.

From (f) we observe that the natural embedding $\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y}_\mathbb{R}) \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ defined by $A(u + iv) := Au + iAv$ is a real-linear isometry, so $\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathtt{Y}_\mathbb{R})$ can be identified with the set of real elements of $\mathcal{B}(\mathtt{U}, \mathtt{Y})$.

Obviously, f is real iff $f = f_R$. Note that $f_R = \operatorname{Re} f$ if f is a constant (operator), e.g., if $f \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$, but for $f(z) = z$ we have $f_R = f \ne \operatorname{Re} f$. Moreover, if $f = ig$ for some real g, then $f_R \equiv 0$ even if g is unbounded or nonholomorphic.

Recall that $f \in \mathrm{L}^\infty_{\mathrm{strong}}$ means that $fu \in \mathrm{L}^\infty$ for every u. We set $\|f\|_{\mathrm{L}^\infty_{\mathrm{strong}}} :=$ $\sup_{u \in \mathtt{U}, \|u\| \le 1} \|fu\|_\infty$. To obtain our Nehari result and some others, we need to establish the following result on $\mathrm{L}^\infty_{\mathrm{strong}}$ (proved in Appendix A).

LEMMA 3.3 ($\mathbf{L^\infty_{strong}}$).

(a) *If* $f : \mathbb{T} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ *is Bochner-measurable, strongly measurable,* L^∞ *or* $\mathrm{L}^\infty_{\mathrm{strong}}$, *then so are* $\bar{f}$, $f(\bar{\cdot})$, $\overline{f(\bar{\cdot})}$, f_R *and* f_I.

(b) *A function* $f \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ *is real iff* fu *is real for all* $u \in \mathtt{U}_\mathbb{R}$.

(c) *Moreover, we have* $\|f_R\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \le \|f\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ *for any* $f \in \mathrm{L}^\infty_{\mathrm{strong}}$.

(d) *However, if* $\dim \mathtt{U} \ge 2$ *and* $\dim \mathtt{Y} \ge 1$, *then there exists a real-symmetric* $f \in$ $\mathrm{L}^\infty_{\mathrm{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ *such that* $\|f\|_{\mathrm{L}^\infty_{\mathrm{strong}}} > \sup_{u = u_R \in \mathtt{U}, \|u\| \le 1} \|fu\|_\infty$.

Claim (d) can be written as $\|f\|_{\mathcal{B}(\mathtt{U}, \mathrm{L}^\infty(\mathtt{Y}))} > \|f\|_{\mathcal{B}(\mathtt{U}_\mathbb{R}, \mathrm{L}^\infty_\mathbb{R}(\mathtt{Y}))}$, where $\mathrm{L}^p_\mathbb{R}$ is the real-symmetric subset of L^p. It can be shown that $\|f\|_{\mathcal{B}(\mathtt{U}, \mathrm{L}^\infty(\mathtt{Y}))} = \|f\|_{\mathcal{B}(\mathrm{L}^2_\mathbb{R}(\mathtt{U}), \mathrm{L}^2_\mathbb{R}(\mathtt{Y}))}$. Claim (b) means, of course, that some function (namely f_R) in the equivalence class of f is real-symmetric if the condition holds.

Thus, we have proved that the Nehari (or Page) Theorem provides a real solution for real functions.

COROLLARY 3.1 (Nehari). *If* $f \in \mathrm{L}^\infty_{\mathrm{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ *is real, then* $\min_{g \in \mathcal{H}^\infty} \|f - g\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ *is achieved by a real* g.

Indeed, if g is minimizing, then so is g_R, because $f = f_R$ and $\|f - g_R\| \le \|f - g_R - ig_I\|$, by Lemma 3.3(c), where g_R, g_I are as in Lemma 3.2(a). The fact that a minimizing g exists, is well known [Pag70, Theorem 4] [Pel03, Theorem 2.2 and p. 70] [Mik07b, Corollary 4.5].

However, this "symmetrization" method does not similarly apply to the Adamjan–Arov–Krein problem (as given by, e.g., [Pel03, Theorem 1.1] or [Mik07b, Theorem

4.6]) for $n > 1$, because, e.g., $f(z) = 1/(1 - iz/2)$ has Hankel rank 1 (since $f(z) = \sum_{k=0}^{\infty}(i/2)^k z^k$), but f_R has Hankel rank 2. We omit the straight-forward details.

We observe that also the real version of *Hartman's Theorem* holds. Indeed, if $f \in L^{\infty}_{\text{strong}}(\mathbb{T}; \mathcal{B}(\mathtt{U}, \mathtt{Y}))$ has a compact "Hankel operator" Γ_f, then $\Gamma_f = \Gamma_g$ for some continuous $g : \mathbb{T} \to \mathcal{BC}(\mathtt{U}, \mathtt{Y})$, where $\mathcal{BC}$ stands for compact operators, by Hartman's Theorem ([Pel03, p. 74] [Pag70, Sections 4&6] [Mik07b, Theorem 4.7]). Moreover, g_R has the same properties if f is real, because then the coefficients $\widehat{f}(n)$ are real, by Lemma 3.2(c), and $\widehat{f}(n) = \widehat{g}(n)$ $(n \geq 1)$, by Hartman's Theorem. As $\widehat{g}(n) = \widehat{g_R}(n) + i\widehat{g_I}(n)$ is real, we have $\widehat{g_R}(n) = \widehat{g}(n) = \widehat{f}(n)$ (for $n \geq 1$). By Theorem 3.3(c) and continuity, $\|g_R\|_{\infty} \leq \|g\|_{\infty}$.

Next we present a standard result on spectral factorization with the additional fact that the factor can be taken real if the original function is real and coercive.

THEOREM 3.4 (Spectral factorization). *Let $F \in \mathcal{H}^{\infty}(\mathcal{B}(\mathtt{U}, \mathtt{Y}))$. If $\epsilon > 0$ and $F^*F \geq \epsilon I$ a.e. on $\mathbb{T}$, then there exists $G \in \mathcal{H}^{\infty}(\mathcal{B}(\mathtt{U}))$ such that $G^{-1} \in \mathcal{H}^{\infty}$ and $F^*F = G^*G$ a.e. on $\mathbb{T}$. If F is real and $\mathtt{U}$ is separable, then we can ensure that G is real too.*

Proof. If $F = fg$ is an inner-outer factorization with $g \in \mathcal{H}^{\infty}(\mathcal{B}(\mathtt{U}, \mathtt{W}))$ for some separable Hilbert space $\mathtt{W}$, then $g^*g = F^*F$ a.e. on $\mathbb{T}$, and we know that g is invertible in $\mathcal{H}^{\infty}$ because of the assumption on F (see, e.g., the proof of [Sta97, Lemma 18], which is based on [RR85]).

Since $g(0)$ is invertible, we have $\dim \mathtt{W} = \dim \mathtt{U} \leq \infty$, so there exists a (unitary) operator $E \in \mathcal{B}(\mathtt{W}, \mathtt{U})$ that maps the fixed basis of $\mathtt{W}$ to that of $\mathtt{U}$. Set $G := Eg \in \mathcal{H}^{\infty}(\mathcal{B}(\mathtt{U}))$ to complete the proof (if F is real, then we can have g (and f) real, by [MS07, Theorem 2.5]; obviously, E is real and hence so is then G). $\square$

However, if $F = i = G$, then $G_R = 0$, so the symmetrization G_R of a solution is not always a solution to $F^*F = G^*G$.

The above separability assumption is unnecessary: the separable case can be extended to the general case by working as in the proof of [Mik09a, Theorem 4.3] (the details are given in [Mik10]).

Also many other standard results on Toeplitz and Hankel operators can be re-proved for the real case, using the tools developed here, as shown in [Mik10, Section 9].

4. Discrete-time systems. We first recall some details on linear, time-invariant discrete-time systems. See, e.g., [Mik02], [OC04], [Sta05] or [Mik09b] for further details.

A *discrete-time system* on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$ is a quadruple $\left(\frac{A\,|\,B}{C\,|\,D}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$. For each (square-summable) *input* (or *control*) $u \in \ell^2(\mathbb{N}; \mathtt{U})$ and *initial state* $x_0 \in \mathtt{X}$, we associate the *state* trajectory $x : \mathbb{N} \to \mathtt{X}$ and *output* $y : \mathbb{N} \to \mathtt{Y}$ through

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ \quad y_k = Cx_k + Du_k, \end{cases} \quad k \in \mathbb{N}. \tag{4.1}$$

The *transfer function* $G := D + C(\cdot^{-1} - A)^{-1}B = D + \cdot C(I - \cdot A)^{-1}B$ of $\left(\frac{A\,|\,B}{C\,|\,D}\right)$ is holomorphic $r^{-1}\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$, where $r^{-1}\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < r\}$ and $r := r(A)$ is the spectral radius of A. We call $\left(\frac{A\,|\,B}{C\,|\,D}\right)$ a *realization* of G. The *Z-transform* $\widehat{u}$ of $u : \mathbb{N} \to \mathtt{U}$ is defined by $\widehat{u}(z) := \sum_n z^n u_n$. For $x_0 = 0$, we have $\widehat{y} = G\widehat{u}$ on $\mathbb{D} \cap r^{-1}\mathbb{D}$ for every $u \in \ell^2(\mathbb{N}; \mathtt{U})$, hence the name "transfer function".

State feedback means that, for some *state-feedback operator* $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$, we use the function $u := Fx + u_{\circlearrowleft}$ as the input, where $u_{\circlearrowleft} : \mathbb{N} \to \mathtt{U}$ denotes an exogenous

input (or disturbance) $u_\circlearrowleft$. Thus, equation (4.1) together with $u = Fx + u_\circlearrowleft$ defines the "closed-loop system" that maps x_0 and $u_\circlearrowleft$ to x and $\begin{bmatrix} y \\ u \end{bmatrix}$. The solution is given by (in place of $\left(\frac{A \mid B}{C \mid D} \right)$)

$$\left(\frac{A + BF \quad\quad\mid\quad B}{\begin{bmatrix} C + DF \\ F \end{bmatrix} \mid\mid \begin{bmatrix} D \\ I \end{bmatrix}} \right), \tag{4.2}$$

The transfer function of the *closed-loop system* (4.2) is obviously given by

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \begin{bmatrix} D \\ I \end{bmatrix} + \begin{bmatrix} C + DF \\ F \end{bmatrix} (z^{-1} - A - BF)^{-1} B. \tag{4.3}$$

From $\widehat{}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N \\ M \end{bmatrix} \widehat{u_\circlearrowleft}$ we conclude that $\hat{y} = NM^{-1}\hat{u}$, i.e., $G = NM^{-1}$. Later we shall see that if F is chosen to be the "LQ-optimal feedback" and $\left(\frac{A \mid B}{C \mid D} \right)$ is real, then N and M are real and weakly coprime. The same holds even if we use the standard normalization. This will lead to a proof of Theorem 2.1.

The system (4.1) is called *output-stable* if $y \in \ell^2$ whenever $x_0 \in \mathtt{X}$ and $u = 0$; *power-stable* if $x \in \ell^2$ whenever $x_0 \in \mathtt{X}$ and $u = 0$. The system (4.1) is called *output-stabilizable* (resp. *power-stabilizable*) if the system (4.3) is output-stable (resp. power-stable) for some $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$.

5. LQ-optimal control. We observe here that the "LQ-optimal" state-feedback operator is real if the system is real, and, consequently, any output- or power-stabilizable system can be output- or power-stabilized by a real state-feedback operator. The proofs will be given in Section 6. We assume that $\left(\frac{A \mid B}{C \mid D} \right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$, as above.

The *LQR problem* (Linear Quadratic Regulator problem) means, given an initial state $x_0 \in \mathtt{X}$, finding $u \in \ell^2$ such that the *LQR cost function* $\|y\|_2^2 + \|u\|_2^2$ is minimized. It is probably the most popular control problem in the literature. In this section we shall now see how the solution of this problem is connected to stabilization by (the LQ-optimal) state feedback.

It is well known that if a system can be formally stabilized, then it can be stabilized by state feedback, as stated in Theorem 5.1 below. By formal stabilization we mean the *Finite Cost Condition*:

$$\text{for each } x_0 \in \mathtt{X} \text{ there exists } u \in \ell^2(\mathbb{N}; \mathtt{U}) \text{ such that } y \in \ell^2. \tag{5.1}$$

If $\left(\frac{A \mid B}{C \mid D} \right)$ are real, then, by linearity, an equivalent condition is:

$$\text{for each real } x_0 \in \mathtt{X} \text{ there exists } u \in \ell^2(\mathbb{N}; \mathtt{U}) \text{ such that } y \in \ell^2. \tag{5.2}$$

We could require the u in (5.2) to be real-valued, by Theorem 5.1 below. By (4.1), then x and y become real too.

THEOREM 5.1. *Assume the Finite Cost Condition* (5.2). *Then there exists a unique* $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ *such that for each* $x_0 \in \mathtt{X}$ *the (state-feedback) input given by* $u_j = F(A + BF)^j x_0$ $(j \in \mathbb{N})$ *strictly minimizes the function* $\|y\|_2^2 + \|u\|_2^2$.

If A, B, C and D are real, then so is F. The functions N and M in (4.3) *are weakly right coprime and F is output-stabilizing.*

(Theorems 5.1 and 5.2 will be proved after Lemma 6.3 below, although only F etc. being real is new.)

The Finite Cost Condition is trivially also necessary to make to function $\|y\|_2^2 + \|u\|_2^2$ finite; moreover, it is equivalent to output-stabilizability. The operator F is called the *LQ-optimal state-feedback operator.*

Thus, if a real system is output-stabilizable, then it is output-stabilizable by a real state-feedback operator (namely the LQ-optimal one), which, in addition, makes the closed-loop transfer functions $\mathcal{H}^\infty$ and weakly right coprime.

If also the dual system $\left(\frac{A^* \mid C^*}{B^* \mid D^*} \right)$ is output-stabilizable, then the functions N and M in Theorem 5.1 are right coprime [CO06].

Theorem 5.1 implies the following (set $C = I$ and $D = 0$ to have $y = x$ and get the claim in parenthesis below).

COROLLARY 5.1 (stabilizing feedback). *Assume that A, B, C and D are real. If the system is output-stabilizable (resp. power stabilizable), then it is output-stabilizable (resp. power stabilizable) by a real state-feedback operator.*

It is well known that the Finite Cost Condition (5.1) can be verified by solving the LQR Riccati equation given below and that the solution of this equation determines the LQ-optimal F.

THEOREM 5.2. *The system $\left(\frac{A \mid B}{C \mid D} \right)$ satisfies the Finite Cost Condition (5.1) iff there exists a nonnegative solution $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ of the* LQR Riccati equation

$$A^* \mathcal{P} A - \mathcal{P} + C^* C \tag{5.3}$$

$$= (C^* D + A^* \mathcal{P} B)(I + D^* D + B^* \mathcal{P} B)^{-1} \tag{5.4}$$

$$(D^* C + B^* \mathcal{P} A). \tag{5.5}$$

Assume (5.1). Then there exists a smallest nonnegative solution $\mathcal{P}_{\min}$ and the LQ-optimal state-feedback $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ is given by

$$S := I + D^* D + B^* \mathcal{P}_{\min} B, \tag{5.6}$$

$$F := -S^{-1}(D^* C + B^* \mathcal{P}_{\min} A). \tag{5.7}$$

Moreover, if A, B, C and D are real, then so are $\mathcal{P}_{\min}$, S and F. Thus, then also $S^{-1/2}$ and the functions $NS^{-1/2}$ and $MS^{-1/2}$ are real; these two functions are also weakly coprime and normalized.

So this provides a real, normalized, weakly coprime factorization of G. Recall from ((4.3)) that $G = NM^{-1}$. Also $G = NM^{-1}$ is a weakly coprime factorization but not necessarily normalized.

Both these factorizations are real if the system is real (Theorem 7.1 below proves that real G do have real realizations). The two functions are actually (strongly) coprime iff G is stabilizable, by Theorems 2.3 and 2.1.

Most of this section can be considered as well known. Indeed, for some less general settings there are LQR and $\mathcal{H}^\infty$ control results for real Hilbert spaces in the literature. For (continuous-time; cf. Section 8 below) Pritchard–Salamon systems such results are given in [vK93]. The fact that the LQ-optimal F determines a weakly coprime factorization was established in [Mik09b]. In the case of finite-dimensional systems this has been well known, because, for rational functions, weak coprimeness is equivalent to coprimeness.

6. Optimal control. In this section we shall prove the results of Section 5 in a more general setting, covering also indefinite cost functions in place of the above "LQR cost function" $\|y\|_2^2 + \|u\|_2^2$. The main result of this section is that in real problems the optimal cost operator is real (and so is the optimal state feedback operator etc.).

In this section we assume that operators $\left(\frac{A\,|\,B}{C\,|\,D}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ and a "cost operator" $J = J^* \in \mathcal{B}(\mathtt{Y})$ are given.

We define the *cost function* (to be optimized) by

$$\mathcal{J}(x_0, u) := \langle y, Jy \rangle_{\ell^2} = \sum_{j=0}^{\infty} \langle y_j, Jy_j \rangle_{\mathtt{Y}} \quad (x_0 \in \mathtt{X}, \ u \in \ell^2(\mathbb{N}; \mathtt{U})). \qquad (6.1)$$

Recall that the output y is defined by (4.1). Thus, if $J = I$, we get $\mathcal{J}(x_0, u) = \|y\|_2^2$. By extending C and D (by, e.g., 0 and I and/or I and 0, respectively), we can add copies of u and/or x to the output. Therefore, the cost (6.1) is very general and covers the LQR cost $\|y\|_2^2 + \|u\|_2^2$ (but (6.1) may also be indefinite).

Given an initial state $x_0 \in \mathtt{X}$, an input $v \in \ell^2(\mathbb{N}; \mathtt{U})$ is called *J-minimal* for x_0 if $\mathcal{J}(x_0, v) \leq \mathcal{J}(x_0, u)$ for every $u \in \ell^2(\mathbb{N}; \mathtt{U})$.

Denote the maps $x_0 \mapsto y$ and $u \mapsto y$ by $\mathscr{C} := CA^{\cdot}$ and $\mathscr{D}$, respectively. Note that

$$(\mathscr{C}x_0)_k = CA^k x_0 \text{ and } (\mathscr{D}u)_k = \sum_{j=0}^{\infty} CA^j B_{k-j-1} + Du_k \text{ for each } k \in \mathbb{N}. \qquad (6.2)$$

Admissible inputs for x_0 are denoted by $\mathcal{U}(x_0) := \{u \in \ell^2(\mathbb{N}; \mathtt{U}) \,|\, y \in \ell^2\}$. An input $u \in \mathcal{U}(x_0)$ is called *J-optimal* for x_0 if $\langle y, J\mathscr{D}\eta \rangle_{\ell^2} = 0$ for each $\eta \in \mathcal{U}(0)$.

One can easily verify that a control is *J*-optimal iff it is a zero of the Fréchet derivative of $\langle y, Jy \rangle_{\ell^2}$ [Mik02, Lemma 8.3.6]. Moreover, if $J \geq 0$, then *J*-optimal and *J*-minimal are equivalent, but in minimax problems a *J*-optimal control can correspond to a saddle point such as the "$\mathcal{H}^{\infty}$ minimax control" [Sta98] [Mik02].

By $\mathcal{U}_{\mathbb{R}}(x_0)$ we denote the set of real elements of $\mathcal{U}(x_0)$. Given a sequence $u : \mathbb{N} \to \mathtt{U}$, by $\operatorname{Re} u := \frac{1}{2}(u + \overline{u})$ we denote the sequence of real parts of u.

We leave the straightforward proof of the following result to the reader.

LEMMA 6.1. *Assume that $\left(\frac{A\,|\,B}{C\,|\,D}\right)$ is real. If $x_1, x_2 \in \mathtt{X}$ are real, then $\mathcal{U}(x_1 + ix_2) = \mathcal{U}(x_1) + i\mathcal{U}(x_2) = \mathcal{U}_{\mathbb{R}}(x_1) + i\mathcal{U}_{\mathbb{R}}(x_2)$ (the set $\mathcal{U}(x_1 + ix_2)$ is empty if any of the other four sets is empty). Moreover, if $x_0 \in \mathtt{X}$ is real and $u \in \mathcal{U}(x_0)$, then $\operatorname{Re} u \in \mathcal{U}_{\mathbb{R}}(x_0)$.*

The following operator is very important in applications. It is usually obtained as the (stabilizing) solution of the Riccati equation corresponding to the problem, which is a generalization of (5.3)–(5.5).

DEFINITION 6.1 ($\mathcal{P}$). *We call $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ the J-optimal cost operator for $\left(\frac{A\,|\,B}{C\,|\,D}\right)$ if, for each $x_0 \in \mathtt{X}$, there exists at least one J-optimal control u with $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle_{\mathtt{X}}$.*

It follows that $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0 \rangle_{\mathtt{X}}$ for every *J*-optimal control u for x_0 [Mik06][2]; in particular, $\mathcal{P}$ is unique.

We can now prove that $\mathcal{P}$ is necessarily real in real problems.

THEOREM 6.2 ($\mathcal{P}$ is real). *Assume that A, B, C, D and J are real. If $x_0 \in \mathtt{X}$ is real and $u \in \ell^2(\mathbb{N}; \mathtt{U})$ is J-optimal for x_0, then $\operatorname{Re} u$ is J-optimal for x_0. Moreover, the J-optimal cost operator, if any, is real.*

Proof. 1° Assume that x_0, u_1 and u_2 are real and $u = u_1 + u_2$. By Lemma 6.1, we have $\mathcal{U}(0) = \{\eta_1 + i\eta_2 \,|\, \eta_1, \eta_2 \in \mathcal{U}_{\mathbb{R}}(0)\}$, so u is *J*-optimal for x_0 iff $\langle y, J\mathscr{D}\eta \rangle = 0$ for each $\eta \in \mathcal{U}_{\mathbb{R}}(0)$, by linearity. But $y = y_1 + iy_2$, where

$$y_1 := \mathscr{C}x_0 + \mathscr{D}u_1, \quad y_2 := \mathscr{D}u_2. \qquad (6.3)$$

---

[2]Actually, [Mik06] treats the continuous-time case but the proof is analogous. Even if the *J*-optimal control were non-unique, the corresponding cost is always unique [Mik06, Lemma 3.5].

Obviously, y_1 and y_2 are real and $\langle y, J\mathscr{D}\eta \rangle = \langle y_1, J\mathscr{D}\eta \rangle + i\langle y_2, J\mathscr{D}\eta \rangle$, hence u is J-optimal for x_0 iff u_1 is J-optimal for x_0 and u_2 is J-optimal for 0.

$2°$ Let $x_1, x_2 \in \mathtt{X}$ be real. If u_k is real and J-optimal for x_k $(k = 1, 2)$, then $\langle x_1, \mathcal{P}x_2 \rangle = \langle \mathscr{C}x_1 + \mathscr{D}u_1, J(\mathscr{C}x_2 + \mathscr{D}u_2) \rangle \in \mathbb{R}$ (expand $\langle x_1 + x_2, \mathcal{P}(x_1 + x_2) \rangle$ to obtain this; use the fact that $u_1 + u_2$ is J-optimal for $x_1 + x_2$). Since x_1 and x_2 were arbitrary, $\mathcal{P}$ is real. $\square$

We call $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ a *J-optimal state-feedback operator* if the corresponding feedback input $k \mapsto F(A + BF)^k x_0$ (i.e., the input $u = Fx$) is J-optimal for x_0, for every $x_0 \in \mathtt{X}$ (see above (4.2)). In real problems, F is real:

LEMMA 6.3 (F is real). *Assume that A, B, C, D and J are real. If F is a J-optimal state-feedback operator and the J-optimal control for 0 is unique, then F is real.*

Proof. Since the J-optimal control for 0 is unique, so is that for any $x_0 \in \mathtt{X}$ (since the difference of two J-optimal controls for x_0 is J-optimal for 0). Let a real $x_0 \in \mathtt{X}$ be given. Then $u := F(A + BF)\dot{}\,x_0$ satisfies $u = \operatorname{Re} u$, by uniqueness and Theorem 6.2, hence u is real, hence $u_0 = Fx_0$ is real. Since x_0 was arbitrary, F is real. $\square$

Proof. [Proof of Theorems 5.1 and 5.2] This follows from [Mik09b, Theorem 1.2 & Proposition 3.1] except that the realness of $\mathcal{P}$ and F and the uniqueness of F are from Theorem 6.2 and Lemma 6.3 (with $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$ and $\mathtt{Y} \times \mathtt{U}$ in place of J, C, D and $\mathtt{Y}$, respectively); by (5.6), also S is real; by (4.3) also N and M are real (also $S^{-1/2}$ is real, by [Chr02, Lemma A.6.7], because $S \geq 0$ and S is real). $\square$

7. Proofs for Section 2. In this section we prove the results of Section 2.

Typical feedback stabilization problems are solvable only for transfer functions that can be factorized as NM^{-1}, where $N, M \in \mathcal{H}^\infty$. Many equivalent characterizations of this "factorizability" are given in [Mik09b, Theorem 1.2].

Here we record the fact that every real "factorizable" function is the transfer function of some real output-stabilizable realization (also the converse holds).

THEOREM 7.1 (realization). *If G is a real proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function and $G = NM^{-1}$ for some $N, M \in \mathcal{H}^\infty$ such that $M(0)$ is invertible, then the shift realization $\left(\begin{smallmatrix} A & | & B \\ \hline C & | & D \end{smallmatrix}\right)$ of G in [Mik09b, Theorem 5.2] is real and output-stabilizable.*

We omit the straightforward proof. The equations $G(z) = N(z)M(z)^{-1}$ and $G(z) = D + C(z^{-1} - A)^{-1}B$ are to hold near the origin. By the realization being real we mean that A, B, C and D are real. Note that N, M are not required to be real. However, often a more suitable realization than the shift realization can be found.

We remark that using the above realization one could always construct a real, stabilizable and detectable realization of G, following the algorithm in [Mik09b, Remark 5.3]).

Proof. [Proof of Theorem 2.1] By Theorem 7.1, NM^{-1} has a real output-stabilizable realization. Theorem 5.2 provides a real normalized "weakly coprime factorization" $N_\mathrm{c}M_\mathrm{c}^{-1}$ of NM^{-1}. The last claim follows from [Mik09b, Theorem 1.1]. $\square$

Proof. [Proof of Theorems 2.3 and 2.2] $1°$ Without the words "real", Theorem 2.3 is contained in [Mik07a, Theorem 1.1].

$2°$ Assume G is real and as in the theorem. The coprime N and M can be taken real, by Theorem 2.1, and so can Y and X, by [MS07]; assume that they are real. Moreover, as in the proof of [Mik07a, Lemma A.5], we can choose the real $X, Y \in \mathcal{H}^\infty$ so that $X(0) = I$ and $Y(0) = 0$.

$3°$ Because $\left[\begin{smallmatrix} M & Y+MV \\ N & X+NV \end{smallmatrix}\right] = \left[\begin{smallmatrix} M & X \\ N & Y \end{smallmatrix}\right]\left[\begin{smallmatrix} I & V \\ 0 & I \end{smallmatrix}\right]$, we observe that $Y + MV$ and $X + NV$ are real iff V is real.

$4°$ Let $V \in \mathcal{H}^\infty$ be such that the $K = Y_1 X_1^{-1}$ in (2.2) is real, where $X_1 := X + NV$ and $Y_1 := Y + MV$. By [Sta05, Theorem 8.5.7], X_1 and Y_1 are coprime. Now $V = V_R + iV_I$, where $V_R, V_I \in \mathcal{H}^\infty$ are real. Moreover,

$$Y + MV_R + iMV_I = Y_1 = KX_1 = KX + KNV_R + iKNV_I. \qquad (7.1)$$

Therefore, $(M - KN)V_I = 0$, hence $V_I = 0$, because $(M - KN) = M^{-1}(I - KNM^{-1}) = M^{-1}(I - KG)$ has a proper inverse [Mik07a, equation (1)]. Thus, V is real.

$5°$ Conversely, if V is real, then so is K, by $3°$, so Theorem 2.3 holds.

$6°$ Take $V = 0$ to observe that Theorem 2.2 holds. □

8. Continuous time results. In this section we prove that the analogies of almost all results of previous sections hold for continuous-time systems too, such as well-posed linear systems (Salamon–Weiss systems). In particular, the unit disc $\mathbb{D}$ is replaced by the right half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ and equation (4.1) is replaced by $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x(0) = x_0$, where A, B and C may be moderately unbounded and D not necessarily well defined. It is often easier to describe the system as $\left[\begin{array}{c|c}\mathscr{A} & \mathscr{B}\tau \\ \hline \mathscr{C} & \mathscr{D}\end{array}\right] : \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$ with the requirements that the system is linear and time-invariant and maps $x_0 \in \mathtt{X}$, $u \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathtt{U})$ boundedly to $x(t) \in \mathtt{X}$, $y \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathtt{Y})$ for some (hence any) $t > 0$. Further details can be found in, e.g., [SW02] [Sta05] [Mik06] [Mik08, Section 5] [Mik02] [WC97].

Most results of Section 2 are obtained for $\mathbb{C}^+$ merely by Cayley transforming, as stated in Remark 8.1(b) below. The standard form of "proper" can also be obtained (see (c) below).

In [Mik06], it was shown that formal output stabilizability (i.e., the Finite Cost Condition) implies stabilizability by well-posed state feedback, by showing that the LQ-optimal state-feedback is well-posed (for parabolic systems this was already known). If the system is real, then the LQ-optimal state-feedback is real too, so any real output-stabilizable (resp., exponentially stabilizable) system is stabilized by well-posed real state feedback (by (e) and (f) below). In the proofs we use the tools developed above, and the same tools can be used to obtain "real" forms of many other standard results too.

REMARK 8.1. **(a)** *A Laplace-transformable function* $f : [0,\infty) \to \mathtt{Z}$ *is (essentially) real-valued iff its Laplace-transform* $\widehat{f}(z) = \int_0^\infty \mathrm{e}^{-tz} f(t)\,dt$ *is real-symmetric.*

(b) *Let* $r > 0$. *The results of Sections 2–3 (except Lemma 3.1((iv)) and Lemma 3.2(b)&(c)) also hold with* $\mathbb{C}^+$, $\mathbb{R}$ *and* r *in place of* $\mathbb{D}$, $\mathbb{T}$ *and* 0, *respectively (in the domains of functions, hence in the definition of* $\mathcal{H}^\infty$, "*proper*", "*coprime*" *etc.)*

(c) *The above result (b) also holds if* "*proper*" *is redefined as* "*defined on some right half-plane*" *(i.e., on* $\{\operatorname{Re} z > \omega\}$ *for some* $\omega \in \mathbb{R}$) *except that in Theorem 2.3 it is not known whether* X^{-1} *can always be taken proper (it can be if, e.g.,* $\lim_{\operatorname{Re} z \to +\infty} G(z)$ *exists).*

(d) *Lemmata 6.1 and 6.3 and Theorem 6.2 also hold if we replace* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *by a linear map* $[\,\mathscr{C} \mid \mathscr{D}\,] : (x_0, u) \mapsto y$ *and* F *(in Lemma 6.3) by any map* $\mathscr{F}_\circlearrowleft$ *such that* $\mathscr{F}_\circlearrowleft(x_0)$ *is* J*-optimal for each* $x_0 \in \mathtt{X}$.

(e) Real version of [Mik06]. *Assume that the map* $[\,\mathscr{C} \mid \mathscr{D}\,]$ *of [Mik06] is real and that the* Finite Cost Condition *holds i.e., for each* $x_0 \in \mathtt{X}$ *there exists* $u \in \mathrm{L}^2([0,\infty);\mathtt{U})$ *such that* $\mathscr{C}x_0 + \mathscr{D}u \in \mathrm{L}^2$ *(we can assume* x_0 *to be real and require* u *to be real, cf. (5.2)).*

Then there exists a real LQ-optimal state-feedback pair $[\,\mathscr{F}_\circlearrowleft\,|\,\mathscr{G}_\circlearrowleft\,]$ such that (in [Mik06]) the corresponding N and M are real, normalized and weakly coprime, $[\,\mathscr{F}\,|\,\mathscr{G}\,]$ are real and $S = I$.

(f) If $[\,\mathscr{A}\,|\,\mathscr{B}\,]$ of [Mik06] are real and the system is exponentially stabilizable, then the system is exponentially stabilizable by a real state-feedback pair.

(It obvious that a real state-feedback pair corresponds to a real state-feedback operator, as defined in, e.g., [Sta05], [Mik02] and [Mik08].)

In Remark 8.1(c), the assumption that $\lim_{\operatorname{Re} z \to +\infty} G(z)$ exists can be replaced by a more general assumption, but a necessary assumption for a proper real stabilizing compensator K to exist is the so-called "parity interlacing condition" (or "positive on real zeros") [Sta92] [Wic10] on *some* right half-plane. For scalar-valued $G = NM^{-1}$, $N, M \in \mathcal{H}^\infty$, this means that M must have the same sign at every zero of N on some right half-axis $\{z > R\}$.

9. Conclusions. In finite-dimensional systems theory, the literature often assumes the data to be real and provides real solutions such as real controllers. Many of these results have recently been generalized to infinite-dimensional systems, but mainly using complex methods and obtaining complex solutions.

In practical applications with real data, real solutions are needed. Similarly, in the development of numerical approximations, one must know a priori that the solutions are real. The solutions being real may also provide to be helpful in the research on further properties of the solutions.

Much of this gap has been covered in above sections. For example, formulas for real solutions for the stabilizing compensator problem and the LQR problem are obtained in Sections 2 and 5, including also real stabilizing state feedback. Real solutions are also provided to many standard tools including coprime and spectral factorizations, Nehari approximation and, also in the general case, the (possibly indefinite or minimax) optimal cost operator, which is the solution of the corresponding (generalized) Riccati equation. Also constructive formulas are outlined.

Moreover, tools and guidelines for developing similar real solutions for further systems theory problems are presented.

For simplicity, Sections 2–7 were written for discrete time, but in Section 8 it was shown that the same results apply in continuous time too, mutatis mutandis.

Appendix A. Proofs of Lemmata 3.2 and 3.3 and Remark 8.1.

Proof. [Proof of Lemma 3.2] (a) Now $f_R(\bar{z}) = \frac{1}{2}[f(\bar{z}) + \overline{f(z)}] = \frac{1}{2}[\overline{f(z)} + f(\bar{z})] = \overline{f_R(z)}$. Moreover, $f_I = \frac{1}{2}(-if + i\overline{f(\cdot)})$, so $f_R + if_I = f$. Now $0 = 0_R + i0_I$, hence $0_R = -i0_I$, so $\overline{0_R(z)} = 0_R(\bar{z}) = -i0_I(\bar{z}) = \overline{i0_I(z)} = \overline{i0_I(z)} = \overline{-0_R(z)}$, hence $0_R(z) = 0 = 0_I(z)$, for every z. Therefore, if $f_R + if_I = g + ih$ for some real g, h, then $f_R - g = 0 = f_I - h$. The rest is clear.

(b) The subsums form convergent series [HP57, p. 97], so this follows from the uniqueness of f_R, f_I (see (a)). (c) is obvious.

(d) We have $f(z)g(z) + \overline{f(\bar{z})}g(\bar{z}) = [f(z) + \overline{f(\bar{z})}]g(z)$.

(e) Now $\langle u + iv, u + iv \rangle = \|u\|^2 + \|v\|^2 + i\langle v, u \rangle - i\langle u, v \rangle = \|u\|^2 + \|v\|^2$, by the standing assumption on $\mathtt{U}_\mathbb{R}$ that $\langle u, v \rangle \in \mathbb{R}$.

(f) Set $T := A_{|_{\mathtt{U}_\mathbb{R}}}$. Now A is isomorphic to $A' := \left[\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right] \in \mathcal{B}(\mathtt{U}_\mathbb{R}^2, \mathtt{Y}_\mathbb{R}^2)$, by (e), and $\|A'\| = \max\{\|T\|, \|\overline{T}\|\} = \|T\|$.

(g) Now $\Omega_2 = \overline{\Omega_2}$, because $g(\bar{z}) = \overline{g(z)}$. Set $F := g^{-1}$. For every $g(z) \in \Omega_2$, we have $F(\overline{g(z)}) = F(g(\bar{z})) = \bar{z} = \overline{F(g(z))}$. Moreover, if h is real, then $h(g(\bar{z})) = h(\overline{g(z)}) = \overline{h(g(z))}$, so $h \circ g$ is real. The converse is analogous, QED. $\square$

Proof. [Proof of Lemma 3.3] (a) The first paragraph is straight-forward. E.g., if, for each $u \in \mathtt{U}$ there exist countably-valued and measurable functions $g_n : \mathbb{T} \to \mathtt{Y}$ ($n \in \mathbb{N}$) such that $g_n \to fu$ a.e. as $n \to +\infty$, then $\bar{g}_n \to \bar{f}\bar{u}$ a.e. Since $\bar{u} \in \mathtt{U}$ was arbitrary, $\bar{f}$ is then strongly measurable. (All operations are well defined: if $fu = gu$ a.e. for each $u \in \mathtt{U}$, then, e.g., $\overline{f}u = \overline{f\bar{u}} = \overline{g\bar{u}} = \overline{g}u$ a.e. for each u.)

(b) If fu is real for all $u \in \mathtt{U}_{\mathbb{R}}$, then $fu = (fu)_R = f_R u$, by Lemma 3.2(d), for $u \in \mathtt{U}_{\mathbb{R}}$, so then $f = f_R$. The converse is obvious.

(c) Assume that $\|f\|_{\mathrm{L}^\infty_{\mathrm{strong}}} < \infty$. By [Mik09a, Proposition 2.2], we can redefine f so that $M := \sup_{z \in \mathbb{T}} \|f(z)\| = \|f\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ (but fu is unchanged a.e., for each $u \in \mathtt{U}$). Now $\|f_R(z)u\| = \frac{1}{2}(\|f(z)u\| + \|\overline{f(\bar{z})}u\|) \leq M\|u\|$ for each $u \in \mathtt{U}$, hence $\|f_R\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \leq M$, QED.

(d) The norm of $f(z) := \begin{bmatrix} 1 - z^2 & 1 + z \end{bmatrix}$ is $2 + 4(\mathrm{Im}\, z)^2$. Set $w := f(i) = \begin{bmatrix} 1 + i & 2 \end{bmatrix}$. Then $f \in \mathrm{L}^\infty(\mathbb{T}; \mathcal{B}(\mathbb{R}^2, \mathbb{R}))$ and $\|f\|_\infty = |f(i)| = \sqrt{6}$, but $|wu| < |w| \cdot |u|$ for $u \in \mathbb{R}^2$, because $w \notin \mathbb{R}^{2 \times 1}$. Therefore, $\sup_{u \in \mathbb{R}^2, |u| \leq 1} \|fu\|_\infty < \sqrt{6}$. $\square$

Proof. [Proof of Remark 8.1] (a) This is straight-forward (use the Laplace inversion formula for "if" [HP57, Theorem 6.3.2]).

(b) The Cayley transform $\phi(z) := (r - z)/(r + z)$ maps $\mathbb{C}^+$ one-to-one and onto $\mathbb{D}$, and $\phi(r) = 0$. It preserves real-symmetricity, by Lemma 3.2(g). Therefore, Theorem 3.4 and the results of Section 2 follow and those of Section 3 arise from same proofs, mutatis mutandis.

(c) This follows from (b) and [Mik08, Theorem 3.1(b)] (and if N, M, M^{-1} are $\mathcal{H}^\infty$ over $\{\mathrm{Re}\, z > \omega\}$ for some $\omega \geq 0$, then all a "weakly coprime factorization" in the r-sense are "weakly coprime factorizations" in the half-plane sense too, and vice versa, for any $r > \omega$).

Assume then that $D := \lim_{\mathrm{Re}\, z \to +\infty} G(z)$ exists. Then, for $F := G - D$ there exists $\begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} T & -U \\ -S & R \end{bmatrix}^{-1} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$ such that $F = NM^{-1} = R^{-1}S$ and that X^{-1} and T^{-1} are proper, by [Mik07a, Theorem 7.4].

Now $\begin{bmatrix} M_G & Y_G \\ N_G & X_G \end{bmatrix} := \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} I & -TY \\ 0 & I \end{bmatrix} \in \mathcal{H}^\infty(\mathcal{B}(\mathtt{U} \times \mathtt{Y}))$ is obviously invertible, $N_G M_G^{-1} = (N + DM)M^{-1} = F + D = G$, and $MT - YS = I$, i.e., $I - MT = YS$. Consequently, $X_G - X = DY - NTY - DMTY = -NTY + D(I - MT)Y = -NTY + DYSY = -FMTY + DYRF$, so $\|X_G(z) - X(z)\| \leq \gamma\|F(z)\| \to 0$, as $\mathrm{Re}\, z \to +\infty$. Therefore, also X_G^{-1} is uniformly bounded for $\mathrm{Re}\, z$ big enough.

(d) This is obvious from the proofs. Note that the other two components of a well-posed linear system, namely $\mathscr{A}$ and $\mathscr{B}$, need not be real and they do not affect $J, \mathcal{P}, S, N, M$ etc.

(e) Now $G := \widehat{\mathscr{D}}$ has a normalized, weakly coprime factorization $N_1 M_1^{-1}$, by [Mik06, Corollary 5.1]. By (b) and Theorem 2.1, G also has a real, normalized, weakly coprime factorization NM^{-1}. By [Mik09b, Theorem 1.1] normalized weakly coprime factorizations are unique modulo a unitary operator, so [Mik06, Lemmata A.5 & 4.4] yield another LQ-optimal pair corresponding to the factorization NM^{-1}.

Also $\mathscr{N}$ and $\mathscr{M}$ obviously are real (i.e., they map real-valued functions to real-valued functions, by (a); or equivalently, $\mathscr{N}$ and $\mathscr{M}$ are real as elements of $\mathcal{B}(\mathrm{L}^2)$, where the basis of L^2 consists of real-valued functions). By (the proof of) [Mik06, Lemmata 4.4], $S = I$. By [Mik06, (2.7)], $\begin{bmatrix} \mathscr{F} & | & \mathscr{G} \end{bmatrix}$ are real.

(f) This follows from (e) and the proof of [Mik06, Corollary 5.4]. (So Corollary 5.1 holds also in continuous-time setting.) □

REFERENCES

[Chr02]   Ole Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston, 2002.

[CO06]   Ruth F. Curtain and Mark R. Opmeer, *Normalized doubly coprime factorizations for infinite-dimensional linear systems*, Math. Control Signals Systems **18** (2006), no. 1, 1–31.

[CO11]   Ruth Curtain and Mark Opmeer, *Coprime factorization and robust stabilization for discrete-time infinite-dimensional systems*, Mathematics of Control, Signals, and Systems (MCSS) **23** (2011), 101–115, 10.1007/s00498-011-0068-5.

[Cur06]   Ruth F. Curtain, *Robustly stabilizing controllers with respect to left-coprime factor perturbations for infinite-dimensional linear systems*, Systems Control Lett. **55** (2006), no. 7, 509–517.

[CWW01]   Ruth F. Curtain, George Weiss, and Martin Weiss, *Stabilization of irrational transfer functions by controllers with internal loop*, Systems, approximation, singular integral operators, and related topics, Oper. Theory Adv. Appl., vol. 129, Birkhäuser, Basel, 2001, pp. 179–207.

[HP57]   Einar Hille and Ralph S. Phillips, *Functional analysis and semi-groups*, revised ed., AMS, Providence, 1957.

[Ino88]   Yujiro Inouye, *Parametrization of compensators for linear systems with transfer functions of bounded type*, Technical report 88-01, Faculty of Eng. Sci., Osaka Univ., Osaka, Japan, 1988.

[Li03]   Bingren Li, *Real operator algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[Mik02]   Kalle M. Mikkola, *Infinite-dimensional linear systems, optimal control and algebraic Riccati equations*, Doctoral dissertation, technical report A452, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2002, `www.math.hut.fi/~kmikkola/research/` .

[Mik06]   ———, *State-feedback stabilization of well-posed linear systems*, Integral Equations Operator Theory **55** (2006), no. 2, 249–271.

[Mik07a]   ———, *Coprime factorization and dynamic stabilization of transfer functions*, SIAM J. Control Optim. **45** (2007), no. 2, 1988–2010.

[Mik07b]   ———, *Hankel and Toeplitz operators on nonseparable Hilbert spaces: further results*, Technical Report A528, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2007, `www.math.hut.fi/reports/` .

[Mik08]   ———, *Weakly coprime factorization and continuous-time systems*, IMA J. Math. Control Inform. **25** (2008), no. 4, 515–546.

[Mik09a]   ———, *Hankel and Toeplitz operators on nonseparable Hilbert spaces*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 1, 109–129.

[Mik09b]   ———, *Weakly coprime factorization and state-feedback stabilization of discrete-time systems*, Math. Control Signals Systems **20** (2009), no. 4, 321–350 pp.

[Mik10]   ———, *Real solutions to control, approximation, factorization, representation, hankel and toeplitz problems*, Technical Report A588, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 2010, `www.math.hut.fi/reports/` .

[MS07]   Kalle M. Mikkola and Amol J. Sasane, *Tolokonnikov's Lemma for real H^∞ and the real disc algebra*, Complex Anal. Oper. Theory **1** (2007), no. 3, 439–446.

[MW09]   Raymond Mortini and Brett D. Wick, *The bass and topological stable ranks of $H_R^\infty(\mathbb{D})$ and $A_R(\mathbb{D})$*, J. Reine Angew. Math. **636** (2009), 1–17.

[OC04]   Mark R. Opmeer and Ruth F. Curtain, *Linear quadratic Gaussian balancing for discrete-time infinite-dimensional linear systems*, SIAM J. Control Optim. **43** (2004), no. 4, 1196–1221.

[Pag70]   Lavon B. Page, *Bounded and compact vectorial Hankel operators*, Trans. Amer. Math. Soc. **150** (1970), 529–539.

[Pel03]   Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer, New York, 2003.

[RR85]   Marvin Rosenblum and James Rovnyak, *Hardy classes and operator theory*, Oxford University Press, New York, Oxford, 1985.

[Smi89]   Malcolm C. Smith, *On stabilization and the existence of coprime factorizations*, IEEE Trans. Automat. Control **34** (1989), no. 9, 1005–1007.

[Sta92]   Olof J. Staffans, *Stabilization of a distributed system with a stable compensator*, Math. Control Signals Systems **5** (1992), no. 1, 1–22.

[Sta97]   ———, *Quadratic optimal control of stable well-posed linear systems*, Trans. Amer. Math. Soc. **349** (1997), 3679–3715.

[Sta98]   ———, *On the distributed stable full information H^∞ minimax problem*, Internat. J. Robust Nonlinear Control **8** (1998), 1255–1305.

[Sta05]   ———, *Well-Posed Linear Systems*, Encyclopedia Math. Appl., vol. 103, Cambridge University Press, Cambridge, 2005.

[SW02]   Olof J. Staffans and George Weiss, *Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup*, Trans. Amer. Math. Soc. **354** (2002), no. 8, 3229–3262.

[Vid85]   Mathukumalli Vidyasagar, *Control system synthesis: a factorization approach*, MIT Press Ser. Signal Process. Optim. Control, 7, MIT Press, Cambridge, MA, 1985.

[vK93]   Bert van Keulen, *H_∞-control for distributed parameter systems: A state space approach*, Birkhäuser Verlag, Basel Boston Berlin, 1993.

[WC97]   George Weiss and Ruth F. Curtain, *Dynamic stabilization of regular linear systems*, IEEE Trans. Autom. Control **42** (1997), 4–21.

[Wic10]   Brett D. Wick, *Stabilization in $H_R^\infty(\mathbb{D})$*, Publ. Mat. **54** (2010), no. 1, 25–52.