INCOMPLETE CORRECTIONS WITH SELFADJOINT OPERATORS

T. EIROLA

To appear in Автоматика и Телемеханика (Automation and Remote Control, in Russian) Т. Эйрола: *Сходимость итерационной процедуры с неполными коррекциями; случай самосопряженных операторов*

Abstract. Incomplete iterations for a linear system with contractive selfadjoint operator is considered. Also the correcting operators are assumed to be selfadjoint. Sufficient conditions for convergence are given and by an example it is shown that the essential part of these conditions cannot be weakened.

Subject classification: Primary 47B, Secondary 65J

Keywords: Incomplete corrections, asynchronous iteration

1. Introduction

Iteration with incomplete corrections usually means recurrence

$$x_{n+1}(\omega) = x_n(\omega) + \lambda_n(\omega)[f(x_n)(\omega) - x_n(\omega)] \qquad \omega \in \Omega$$

for finding a fixed point of f, i.e., $x = f(x)$ in a space of some functions $\Omega \to E$. Here λ_n is a real function with values in $[0, 1]$. Incomplete means that only values $x_n(\omega)$ for which $\lambda_n(\omega) > 0$ are corrected.
Such iterations appear for example in the study of desynchronized systems, estimation with incomplete information, and as models of asynchronous parallel computation of fixed points (see: [1-8]). Typical conditions for convergence require some type of contractivity of f together with uniform divergence of the sum $\sum_n \lambda_n$ of the "diagonal" operators λ_n.
In this work we study linear equations

$$x = Ax + b \; . \tag{1.1}$$

In the case of (1.1) in $\mathbf{R}^d$ it is known (see [4]) that the iteration

$$x_{n+1} = x_n + \Lambda_n(Ax_n + b - x_n) \tag{1.2}$$

(where $\Lambda_n = \text{diag}[\lambda_{n1}, \dots, \lambda_{nd}]$) converges if the spectral radius of the matrix

$$|A| := \{|A_{i,j}|\}_{i,j=1}^d$$

1991 *Mathematics Subject Classification.* Primary 47B, Secondary 65J.
Key words and phrases. Incomplete corrections, asynchronous iteration.

is less than one, provided that

$$\sum_{n\geq 0} \lambda_{ni} = \infty \qquad \text{for all } i = 1, \ldots, d\,. \tag{1.3}$$

This paper considers linear equation (1.1) in a real Hilbert space assuming that A is selfadjoint and has spectral radius less than one. Also in the place of diagonal multipliers we assume more generally that operators D_n are selfadjoint with spectrum in $[0,1]$. Hence our topic is the iteration

$$x_{n+1} = x_n + D_n(Ax_n + b - x_n)\ , \tag{1.4}$$

in a Hilbert space H assuming

$$A\,,\ D_n\,,\ n \geq 1 \quad \text{are selfadjoint and} \quad \rho(A) < 1\,,\ \sigma(D_n) \subset [0,1]\,. \tag{1.5}$$

Remark 1.1. Note that in the case of matrices

$$A \quad \text{selfadjoint and} \quad \rho(A) < 1 \;\not\Rightarrow\; \rho(|A|) < 1$$

as $A = \frac{1}{6}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$ shows: $\rho(A) = 5/6\,,\ \rho(|A|) = 7/6\,.$

In section 2 we prove that if the D_n–operators "touch everything" with the rate of touching not tending to zero too fast (see conditions i) and ii) of Theorem 2.1), then the iteration (1.4) converges to the solution of (1.1).
Further, in section 3 we construct an example, which shows that the conditions of the theorem cannot be essentially weakened.

2. The main result

The following theorem gives sufficient conditions for the convergence of the iteration (1.4).
The idea of conditions i) and ii) is to prevent the error vector turning away without shortening enough before it is hit. Quantitatively this is then checked in claims 2.3 and 2.4.

Theorem 2.1. *Suppose conditions* (1.5) *hold. Assume there exists an increasing sequence* $\{n_j\}_{j=1}^{\infty}$ *of integers and a sequence* $\{\delta_j\}_{j=1}^{\infty}$ *of nonnegative numbers such that*

i) for all x and j holds: $\max\limits_{n_j \leq i < n_{j+1}} \langle D_i x, x\rangle \geq \delta_j\, |x|^2$

ii) $\displaystyle\sum_{j=1}^{\infty} \frac{\delta_j}{n_{j+1} - n_j}$ diverges

Then the iteration (1.4) *converges to the solution of* (1.1).

Proof. Set $\varepsilon = 1 - \|A\|$ and let x^* be the solution of (1.1). Set

$$u_n = (I - A)^{\frac{1}{2}}(x_n - x^*)\ .$$

Then

$$\begin{aligned} u_{n+1} &= (I-A)^{\frac{1}{2}}\Big[x_n + D_n(Ax_n + b - x_n - Ax^* - b + x^*) - x^*\Big] = \\ &= u_n - (I-A)^{\frac{1}{2}} D_n (I-A)(x_n - x^*) = (I - B_n)u_n\ , \end{aligned}$$

where $B_n := (I-A)^{\frac{1}{2}} D_n (I-A)^{\frac{1}{2}}$ is selfadjoint and $\sigma(B_n) \subset [0, 2-\varepsilon]$. Note that $x_n \to x^*$ is equivalent to $u_n \to 0$ and that the sequence $\{|u_n|\}$ is nonincreasing.

Lemma 2.2. *If $k \geq n$ and $\langle B_k u_n, u_n \rangle = \delta |u_n|^2$, then*

$$|u_{k+1}|^2 \leq e^{-\frac{\varepsilon\delta}{2\pi(k-n)+\pi}} \; |u_n|^2 \; . \tag{2.1}$$

Assume this lemma is true. Since $\left|(I-A)^{\frac{1}{2}} u\right| \geq \sqrt{\varepsilon}\,|u|$ and

$$\langle B_i u, u \rangle = \left\langle D_i (I-A)^{\frac{1}{2}} u, (I-A)^{\frac{1}{2}} u \right\rangle$$

we have

$$\max_{n_j \leq i < n_{j+1}} \langle B_i u, u \rangle \geq \varepsilon \delta_j \, |u|^2 \; .$$

If $k \in [n_j, n_{j+1})$ is such that $\left\langle B_k u_{n_j}, u_{n_j} \right\rangle \geq \varepsilon\delta_j \left|u_{n_j}\right|^2$, then by lemma 2.2

$$\left|u_{n_{j+1}}\right|^2 \leq |u_{k+1}|^2 \leq e^{-\frac{\varepsilon^2\delta_j}{2\pi(k-n_j)+\pi}} \; \left|u_{n_j}\right|^2 \leq e^{-\frac{\varepsilon^2\delta_j}{2\pi(n_{j+1}-n_j)+\pi}} \; \left|u_{n_j}\right|^2 \; .$$

Then

$$|u_{n_m}|^2 \leq \prod_{j=1}^{m} e^{\frac{-\,\varepsilon^2\delta_j}{2\pi(n_{j+1}-n_j)+\pi}} \, |u_{n_1}|^2 \to 0 \qquad \text{as} \quad m \to \infty$$

because $\sum_{j=1}^{\infty} \frac{\varepsilon^2\delta_j}{2\pi(n_{j+1}-n_j)+\pi} \geq \frac{\varepsilon^2}{3\pi} \sum_{j=1}^{\infty} \frac{\delta_j}{n_{j+1}-n_j} = \infty$. □

Proof of Lemma 2.2. Set $\rho = 1 - \frac{\varepsilon}{2}$, $j = k-n$, and let $\phi \geq 0$ be the angle between u_k and u_n . We begin with the following two claims:

Claim 2.3.

$$|u_k| \leq \eta(\phi/j)^j \, |u_n| \; , \tag{2.2}$$

where $\eta(t) := \frac{\varepsilon}{2}\cos(t) + \sqrt{1-\varepsilon+(\frac{\varepsilon}{2}\cos(t))^2}$.

Claim 2.4. *If* $\sin(\phi) \leq \sin(\psi) := \sqrt{\frac{\delta}{2\rho}}$, *then*

$$|(I-B_k)u_k|^2 \leq \left(1 - 2\varepsilon\rho\; \sin^2(\psi-\phi)\right) |u_k|^2 \; . \tag{2.3}$$

Proof of Claim 2.3. We show first that if the angle between u_{i+1} and u_i is τ_i , then

$$|u_{i+1}| \leq \eta(\tau_i) \, |u_i| \; . \tag{2.4}$$

For this we note that the spectrum of $\rho I - B_i$ lies in $[-\rho, \rho]$, which implies

$$\left|u_{i+1} - \frac{\varepsilon}{2} u_i\right| = |\rho u_i - B_i u_i| \leq (1 - \frac{\varepsilon}{2}) \, |u_i| \; . \tag{2.5}$$

Thus

$$|u_{i+1}|^2 - \varepsilon\cos(\tau_i)\, |u_{i+1}|\, |u_i| \leq (1-\varepsilon)\, |u_i|^2$$

and

$$|u_{i+1}| \leq \left(\tfrac{\varepsilon}{2}\cos(\tau_i) + \sqrt{1-\varepsilon+(\tfrac{\varepsilon}{2}\cos(\tau_i))^2} \right) |u_i| \; ,$$

which is (2.4). Now for (2.2) we want to find the maximum of $\prod_{i=n}^{k-1} \eta(\tau_i)$ subject to $\sum_i \tau_i \geq \phi$. Set $f(\tau) = \log(\eta(\tau))$. Denoting: $s(t) = \sqrt{1 - \varepsilon + (\frac{\varepsilon}{2}\cos(t))^2}$ we find that

$$f'(t) = -\frac{\varepsilon \sin t}{s(t)} \quad \text{and} \quad f''(t) = -\frac{\varepsilon(2-\varepsilon)^2 \cos t}{s(t)^3}$$

are both negative for $t \in (0, \pi/2)$. Hence if $\tau_i \in [0, \pi/2]$ and $\sum_i \tau_i \geq \phi$, then

$$\begin{aligned} \sum_{i=n}^{k-1} f(\tau_i) &\leq \sum_{i=n}^{k-1} [f(\phi/j) + f'(\phi/j)(\tau_i - \phi/j)] = \\ &= j f(\phi/j) + f'(\phi/j)\Big(\sum_i \tau_i - \phi\Big) \leq j f(\phi/j) \ . \end{aligned} \tag{2.6}$$

Note that the maximising τ_i's cannot have some $\tau_i > \pi/2$, since if $\tau \in (\pi/2, \phi)$, then $\frac{\phi-\tau}{j-1} \leq \phi - \tau \leq \pi - \tau < \frac{\pi}{2}$ so that

$$\tfrac{d}{d\tau}[f(\tau) + (j-1) f(\tfrac{\phi-\tau}{j-1})] = f'(\tau) - f'(\tfrac{\phi-\tau}{j-1}) \leq f'(\tau) - f'(\pi - \tau) = 0$$

with strict inequality unless $\phi = \pi$ and $j = 2$ in which case claim 2.3 follows directly from (2.4).

Now (2.6) implies $\displaystyle\prod_{i=n}^{k-1} \eta(\tau_i) \leq \eta(\phi/j)^j$ and the claim follows. $\square$

Proof of Claim 2.4. Let v be the unit vector in the direction of u_k i.e.,

$$v = (c u_n + s u_n') / |u_n| \ ,$$

where $c = \cos(\phi)$, $s = \sin(\phi)$ and $u_n' \perp u_n$. We have

$$\begin{aligned} |u_{k+1}|^2 &= |u_k|^2 - 2\langle B_k u_k, u_k \rangle + |B_k u_k|^2 = \\ &= |u_k|^2 - \varepsilon \langle B_k u_k, u_k \rangle + \langle B_k (B_k - 2 + \varepsilon) u_k, u_k \rangle \leq \\ &\leq (1 - \varepsilon \langle B_k v, v \rangle) |u_k|^2 \ . \end{aligned} \tag{2.7}$$

Thus we want to find a lower bound for

$$g(\gamma, \beta) := \langle B_n v, v \rangle = c^2 \delta + 2cs\gamma + s^2 \beta = \begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} \delta & \gamma \\ \gamma & \beta \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} \ ,$$

where

$$\gamma := \langle B_k u_n, u_n' \rangle / |u_n|^2 \quad \text{and} \quad \beta := \langle B_k u_n', u_n' \rangle / |u_n|^2 \ .$$

Because the eigenvalues of $\begin{bmatrix} \delta & \gamma \\ \gamma & \beta \end{bmatrix}$ are in $[0, 2\rho]$, we have the constraints

$$\gamma^2 \leq \delta\beta \quad \text{and} \quad \gamma^2 \leq (2\rho - \delta)(2\rho - \beta) \ .$$

Thus we want to find the minimum of $g(\gamma, \beta)$ in the convex set

$$\gamma^2/\delta \ \leq \ \beta \ \leq \ 2\rho - \gamma^2/(2\rho - \delta) \ , \quad \gamma^2 \ \leq \ \delta(2\rho - \delta) \ .$$

Since g is linear in β we get

$$\min_\beta g(\gamma, \beta) = g(\gamma, \gamma^2/\delta) = \frac{s^2}{\delta}(\frac{c}{s}\delta + \gamma)^2$$

and find that if

$$\frac{c^2}{s^2}\delta^2 \geq \delta(2\rho-\delta)\ , \text{ i.e., if } \quad \sin(\phi) \leq \sqrt{\tfrac{\delta}{2\rho}} = \sin(\psi)\ ,$$

then the minimum is obtained at $\gamma = -\sqrt{\delta(2\rho-\delta)}$ and we have

$$\langle B_k v, v\rangle \geq \Big(c\sqrt{\delta} - s\sqrt{2\rho-\delta}\ \Big)^2 = 2\rho\ \sin^2(\psi-\phi)\ ,$$

which together with (2.7) implies (2.3). (Note that if $\sin(\phi) \geq \sqrt{\delta/2\rho}$, then we can have $\langle B_k v, v\rangle = 0$.) □

Now back to the proof of Lemma 2.2. From the two claims we get:

$$|u_{k+1}|^2 \leq \max_{0\leq\pi\leq\psi} \zeta(\phi)\ |u_n|^2 = \max_{0\leq\phi\leq\psi} \zeta(\phi)\ |u_n|^2\ ,$$

where

$$\zeta(\phi) := \eta(\phi/j)^{2j}\left(1-2\varepsilon\rho\ \sin^2(\psi-\phi)\right)\ .$$

One gets easily:

$$\eta(t) \leq \frac{\varepsilon}{2}\cos(t) + \sqrt{1-\varepsilon+(\frac{\varepsilon}{2})^2} = 1-\tfrac{\varepsilon}{2}(1-\cos(t))$$

and further, using $\log(1-x) \leq -x$

$$\zeta(\phi) \leq e^{-\varepsilon\,[j(1-\cos(\phi/j)) + 2\rho\sin^2(\psi-\phi)]}\ .$$

With inequalities $\cos(t) \leq 1-t^2/\pi$ and $\sin(t) \geq 2t/\pi$ for $t\in[0,\pi/2]$ we get

$$\zeta(\phi) \leq e^{-\varepsilon\,[\frac{\phi^2}{\pi j}+\frac{8\rho}{\pi^2}(\psi-\phi)^2]} \leq e^{-\frac{8\varepsilon\rho}{\pi^2+8\pi\rho j}\arcsin\left(\sqrt{\delta/2\rho}\right)^2} \leq$$
$$\leq e^{-\frac{4\varepsilon\delta}{\pi^2+8\pi\rho j}} \leq e^{-\frac{\varepsilon\delta}{\pi+2\pi\rho j}}\ ,$$

from which (2.1) follows. □

Since

$$\langle D_i x, x\rangle = |D_i x|^2 + \langle (I-D_i)D_i x, x\rangle \geq |D_i x|^2$$

we also get

Corollary 2.5. *Assume there exists an increasing sequence $\{n_j\}$ of integers and a sequence $\{\mu_j\}$ of nonnegative numbers such that*

i) *for all x and j holds* $\max_{n_j\leq i<n_{j+1}} |D_i x| \geq \mu_j\,|x|$

ii) $\sum_j \frac{\mu_j^2}{n_{j+1}-n_j}$ *diverges*

Then (1.4) *converges.*

3. Example

Here we show by an example that condition ii) of the theorem cannot be replaced by a weaker form

$$\sum_j \frac{\delta_j}{(n_{j+1}-n_j)^\gamma} \quad \text{diverges}$$

with any $\gamma \in [0,1)$. The idea is to use projectors D_n the kernels of which rotate with a decreasing speed.

Example 3.1. *Let $H = \mathbf{R}^2$ and $A = 0$, $b = 0$. Hence we solve $x = 0$ by iteration*

$$x_{n+1} = x_n - D_n x_n\,. \tag{3.1}$$

Using the notation of the proof of the theorem we have now $B_n = D_n$ and $u_n = x_n$.

For given $\gamma \in [0,1)$ we will construct x_0, increasing n_j's, and matrices D_i (which will be projectors), such that

$$\max_{n_j \le i < n_{j+1}} \langle D_i, x, x\rangle \ge \frac{1}{2}|x|^2 \quad \textit{for all} \quad x \in \mathbf{R}^2\,,\ j \ge 0\,, \tag{3.2}$$

$$\frac{1}{2}\sum_j \frac{1}{(n_{j+1}-n_j)^\gamma} \quad \textit{diverges, and} \tag{3.3}$$

$$\inf_n |u_n| > 0\,, \quad \textit{i.e.,} \quad x_n \not\to x^*\,. \tag{3.4}$$

Construction: Given $\phi \in [0, \pi/2]$, let

$$D_\phi = \begin{bmatrix} \sin^2\phi & \sin\phi\,\cos\phi \\ \sin\phi\,\cos\phi & \cos^2\phi \end{bmatrix}.$$

Then $\sigma(D_n) = \{0,1\}$ and if $u = \begin{bmatrix}1\\0\end{bmatrix}$, we have

$$u - D_\phi u = \begin{bmatrix} \cos^2\phi \\ \sin\phi\,\cos\phi \end{bmatrix}, \quad \text{i.e.,} \quad |u - D_\phi u|^2 = \cos^2\phi$$

and the angle between u and $u - D_\phi u$ is ϕ. Hence $I - D_\phi$ is a projector that turns u by angle ϕ.

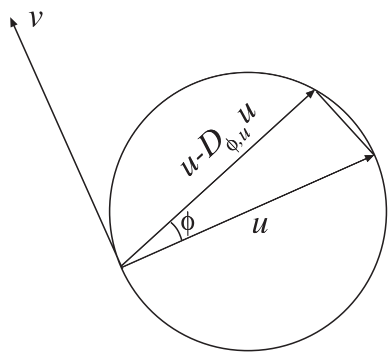
For general $u = \begin{bmatrix}u^1\\u^2\end{bmatrix}$, let $v = \begin{bmatrix}-u^2\\u^1\end{bmatrix}$, and

$$D_{\phi,u} u = \sin^2\!\phi\ u + \sin\phi\,\cos\phi\ v$$
$$D_{\phi,u} v = \sin\phi\,\cos\phi\ u + \cos^2\!\phi\ v\,.$$

Then $I - D_{\phi,u}$ is a projection such that

$$|u - D_{\phi,u} u| = \cos\phi\,|u|\ ,$$
$$\langle u - D_{\phi,u}u, u\rangle = \cos\phi\,|u - D_{\phi,u}u|\,|u|\ ,$$

i.e., $I - D_{\phi,u}$ shortens u by $\cos\phi$ and turns it by ϕ in the positive direction.



For $\gamma \in [0,1)$, let $\{m_j\}_{j=1}^{\infty} \subset \mathbf{N}$ be such that $m_j \geq 2$, and

$$\sum_j m_j^{-1} \quad \text{converges and} \quad \sum_j m_j^{-\gamma} \quad \text{diverges} \tag{3.5}$$

(e.g. $m_j = \left\lceil (1+j)^{1/\gamma} \right\rceil$). Put

$$n_1 = 0\,, \quad n_2 = n_1 + 2m_1\,, \quad k_1 = n_1 + m_1\,, \quad \text{and} \quad \phi_1 = \frac{\pi}{2m_1}\,.$$

Take

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,, \quad D_0 = D_{\phi_1, u_0}\,, \quad u_1 = u_0 - D_0 u_0$$

$$D_i = D_{\phi_1, u_i}\,, \quad u_{i+1} = u_i - D_i u_i\,, \quad i = n_1, \ldots, k_1 - 1\,.$$

Then $u_{k_1} = (\cos\phi_1)^{m_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $D_{k_1 - 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Continue by

$$D_i = D_{\phi_1, u_i}\,, \quad u_{i+1} = u_i - D_i u_i\,, \quad i = k_1, \ldots, n_2 - 1$$

and get $u_{n_2} = -(\cos\phi_1)^{2m_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $D_{n_2-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then we have

$$u_{n_2} = -(\cos\tfrac{\pi}{2m_1})^{2m_1} u_{n_1}\,, \quad n_2 - n_1 = 2m_1\,,$$

and

$$\max_{n_1 \leq i < n_2} \langle D_i, x, x\rangle \geq \max\left(\langle D_{k_1}, x, x\rangle, \langle D_{l_1}, x, x\rangle\right) \geq \frac{1}{2}|x|^2\,.$$

Proceeding this way with: $n_{j+1} = n_j + 2m_j$, $\phi_j = \frac{\pi}{2m_j}$, gives

$$u_{n_{j+1}} = -(\cos\tfrac{\pi}{2m_j})^{2m_j} u_{n_j}\,, \quad n_{j+1} - n_j = 2m_j\,,$$

and

$$\max_{n_j \leq i < n_{j+1}} \langle D_i, x, x\rangle \geq \frac{1}{2}|x|^2\,.$$

Using

$$\cos t \geq 1 - t^2/2 \qquad \text{and} \qquad \log(1-t) \geq \frac{t}{c}\log(1-c) \quad \text{for } 0 \leq t \leq c$$

(with $c = \frac{\pi^2}{32}$) we obtain

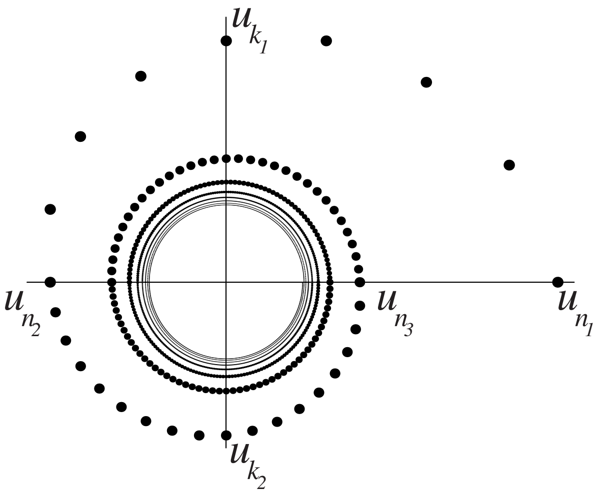
$$\begin{aligned} \left|u_{n_{j+1}}\right| &= \prod_{k=1}^{j} (\cos\tfrac{\pi}{2m_k})^{2m_k} \geq \prod_{k=1}^{j}(1 - \tfrac{\pi^2}{8m_k^2})^{2m_k} = \\ &= e^{\sum_{k=1}^{j} 2m_k \log(1 - \frac{\pi^2}{8m_k^2})} \geq \\ &\geq e^{\sum_{k=1}^{j} \frac{8}{m_k}\log(1-c)} \geq (1-c)^{\sum_{k=1}^{\infty} 8/m_k} > 0\,. \end{aligned}$$

On the other hand,

$$\sum_{j\geq 1}(n_{j+1} - n_j)^{-\gamma} = \sum_{j \geq 1}(2m_j)^{-\gamma} = \infty\,.$$

Hence all the conditions (3.2), (3.3), and (3.4) are satisfied.

The following illustrates the sequence $\{u_i\}_{i\geq 0}$ for $m_j = (1+j)^2 = 4,\, 9,\, 16,\, 25, \ldots$



Remark 3.1. An interesting situation not considered here is the case of (1.2) when A is selfadjoint. In another work (see: [5]) it will be shown that then $\rho(A) < 1$ – instead of $\rho(|A|) < 1$ – suffices for convergence, provided that (1.3) holds.

References

1. E. A. Asarin, M. A. Krasnosel'skii, V. S.Kozyakin, and N. A. Kuznetsov, *Stability analysis of desynchronized systems*, Nauka, Moscow, 1992, In Russian.
2. I. N. Beletskii, *Multiprocerros and parallel structures with organization of asynchronous computations*, Naukova Dumka, Kiev, 1988, In Russian.
3. D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and distributed computation. numerical methods*, Prentice-Hall, New Jersey, 1987.
4. D. Chazan and W. Miranker, *Chaotic relaxation*, Linear Algebra and Appl. **2** (1969), 199–222.
5. T. Eirola, *Random relaxation: the linear selfadjoint case*, to appear (1995).
6. T. Eirola, A. M. Krasnosel'skii, M. A. Krasnosel'skii, N. A. Kuznetsov, and O. Nevanlinna, *Incomplete corrections in nonlinear problems*, Helsinki Univ. Techn. Math. Report A318 (1993), to appear in Nonlin. Anal. Theory, Meth. Appl.
7. A. M. Krasnosel'skii, M. A. Krasnosel'skii, and N. A. Kuznetsov, *Nonlinear systems with incomplete correction and their stability*, Doklady Acad. Nauk **318** (1991), no. 3, 291–294.
8. F. Robert, M. Charnay, and F. Musy, *Iterations chaotiques serie–parallele pour des equations non–lineaires de point fixe*, Aplikace Matematiky **20** (1975), no. 1, 1–38.

Institute of Mathematics, Helsinki University of Technology

FIN-02150 Espoo, Finland
E-mail address: Timo.Eirola@hut.fi