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PSEUDOTRAJECTORIES GENERATED BY A DISCRETIZATION OF A PARABOLIC EQUATION

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Abstract. A semi-implicit discretization of a parabolic equation is considered. The resulting diffeomorphism is shown to be generically Morse–Smale. Uniform bounds for the dimension of its attractor are given and numerical trajectories — including round–off errors — are shown to approximate the attractor.

1. Introduction

A lot of attention has recently been paid to the numerical investigation of attractors of evolution systems generated by partial differential equations. In practice common schemes are expected to perform well. In (Hale, (1994)) J. Hale posed the problem of "understanding the influence of the error terms".

In this paper we consider a full discretization of a parabolic PDE including round–off errors and show how to apply results of the theory of pseudotrajectories (see: Pilyugin, (1994a,b)). We study a finite dimensional diffeomorphism generated by a semi–implicit discretization of a one–dimensional reaction–diffusion equation and show that

(i) it is dissipative;
(ii) the Hausdorff dimension of its global attractor is bounded uniformly with respect to the step sizes;
(iii) for a generic non–linear term of the equation it is a Morse–Smale diffeomorphism.

Applying results of (Pilyugin, (1994a)), we then show that the numerical "attractors" approximate (in the Hausdorff–distance) the real attractor of our diffeomorphism. Future work will study the relations between these and the attractors of the exact evolution system.

We consider a parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)\,, \tag{1}$$

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where $x \in [0,1]$, $t > 0$, with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ and with initial value $u(0,x) = u_0(x)$. It is well-known that under reasonable conditions on f the semigroup $S(t)$ generated by (1) in $X = L^2[0,1]$ has a global attractor I (see: Babin and Vishik, (1992), Hale, (1988), Ladyzhenskaya, (1991a)) that is a compact positively invariant set such that for any u_0 we have $S(t)u_0 \to I$ as $t \to \infty$.

We consider the following semi-implicit discretization of (1). Fix a natural N, set $d = \frac{1}{N+1}$, and let $h > 0$ be the time step. We approximate the values $u(md, nh)$ of a solution of (1) by v_m^n, $n \geq 0$, $m = 0, \ldots, N+1$, given by the system of equations

$$\Delta v^{n+1} = A v^{n+1} + \underline{f}(v^n), \quad n \geq 0, \tag{2}$$

where

$$v^n = (v_1^n, \ldots, v_N^n) \in I\!R^N, \quad \underline{f}(v) = (f(v_1), \ldots, f(v_N)),$$

and

$$\Delta v^{n+1} = \tfrac{1}{h}(v^{n+1} - v^n), \quad (Av)_m = \tfrac{1}{d^2}\left[v_{m+1} - 2v_m + v_{m-1}\right],$$

with $v_0 = v_{N+1} = 0$. This system defines a map $\phi = \phi_{h,N} : I\!R^N \to I\!R^N$ such that $v^{n+1} = \phi(v^n)$, by

$$\phi(v) = J^{-1}(v + h\underline{f}(v)), \tag{3}$$

where $J = E - hA$, E being the unit matrix.

System (2) was investigated in (Oliva et al., (1994)). It was shown there that if

$$f \in C^1, \quad |f'(u)| \leq M, \quad hM < 1, \tag{4}$$

then ϕ is a diffeomorphism of class C^1. We assume everywhere in this paper that conditions (4) are satisfied.

Due to roundoff errors we get from a computer not the sequence v^n generated by (2) but a sequence $w^n \in I\!R^N$, $n \geq 0$, satisfying

$$\|\Delta w^{n+1} - A w^{n+1} - \underline{f}(w^n)\| < \delta, \ n \geq 0, \tag{5}$$

with some $\delta > 0$.

In this paper we study the problem of approximation of attractors $I_{h,N}$ of diffeomorphisms $\phi_{h,N}$ by sequences w^n, which satisfy (5).

Recall the definition of a pseudotrajectory (Conley, (1978), Pilyugin, (1994b)). Let ψ be a dynamical system with phase space $\mathcal{D}$, and let d be a metric on $\mathcal{D}$. Fix $\delta > 0$. A sequence $z^n \in \mathcal{D}$, $n \geq 0$, is called a δ-trajectory (a pseudotrajectory) of ψ if

$$d(z^{n+1}, \psi(z^n)) < \delta, \ n \geq 0.$$

We show below (Lemma 2) that a sequence w^n satisfying (5) is indeed an $l\delta$-trajectory of $\phi_{h,N}$ with $l = l(h,N)$.

In section 2 we establish some other properties of $\phi_{h,N}$ that we need. It is shown that under suitable conditions:

(i) $\phi_{h,N}$ is dissipative (Theorem 1);
(ii) for compact invariant sets $I_{h,N}$ of $\phi_{h,N}$ holds $\sup_{h,N} \dim_H(I_{h,N}) < \infty$, where $\dim_H$ is the Hausdorff dimension (Theorem 2);
(iii) for a C^p-generic f, with $p \geq 1$, $\phi_{h,N}$ is a Morse–Smale diffeomorphism (Theorem 4).

Remark 1. *For other discretizations of* (1)*, analogs of Theorem* 1 *(dissipativity) were obtained in (Humphries et al., (1993) and Ladyzhenskaya, (1991b)) and of Theorem* 2 *(dimension estimates) in (Ladyzhenskaya, (1991b)).*
By results of (Angenent, (1986), Brunovsky and Chow, (1984), and Henry, (1985)) for a C^p–generic f, $p \geq 2$, the semigroup $S(t)$ has the Morse–Smale property.

In section 3 we apply results of the paper (Pilyugin, (1994a)) in which approximation of attractors of finite-dimensional dynamical systems by pseudotrajectories was investigated.

2. Properties of $\phi_{h,N}$

In (Oliva et al., (1994)) the following properties of ϕ were established under assumptions (4).

Theorem 0. *The nonwandering set Ω of ϕ consists of fixed points and if p and q are hyperbolic fixed points of ϕ, then the stable manifold of p is transversal to the unstable manifold of q.*

Let us begin with some standard definitions. For vectors $v,\, w \in I\!R^N$ let

$$\langle v, w\rangle = d \sum_{m=1}^{N} u_m v_m\,, \ \|v\|^2 = \langle v, v\rangle \ .$$

We say that a diffeomorphism $\phi : I\!R^N \to I\!R^N$ is *dissipative* (in the sense of Levinson) if there exists a bounded set $B \subset I\!R^N$ such that for any $v^0 \in I\!R^N$ there exists n_0 with the property:

$$\phi^n(v^0) \in B\,, \ n \geq n_0\,.$$

We say that $I \subset I\!R^N$ is an *attractor* of ϕ, if:

1) I is compact and ϕ–invariant;
2) I is Lyapunov stable, that is given a neighborhood U of I there exists a neighborhood V of I such that

$$\phi^k(V) \subset U \quad \text{for} \quad k \geq 0\,;$$

3) there is a neighborhood W of I such that for $v \in W$

$$\lim_{k\to\infty} d(\phi^k(v), I) = 0\,. \tag{6}$$

For an attractor I let $D(I)$ be its basin:

$$D(I) = \left\{ v \in I\!R^N \,\middle|\, (6) \text{ holds} \right\} .$$

We say that I is a global attractor of ϕ if $D(I) = I\!R^N$. It is well-known (see: Levinson, (1944)) that any dissipative ϕ has a global attractor $I \subset B$.

Let $\mathcal{B}_{a_0,a_1}$ be the set of functions $f \in C^1(I\!R)$, such that for any $u \in I\!R$

$$uf(u) \le a_0 + a_1 u^2 , \tag{7}$$

which further implies that for any $N \ge 1$, $v \in I\!R^N$ holds

$$\left\langle \underline{f}(v), v \right\rangle \le a_0 + a_1 \|v\|^2 . \tag{8}$$

Theorem 1. *Assume that $f \in \mathcal{B}_{a_0,a_1}$ with $a_1 < \pi^2$. Then there exist h_0, N_0, and $\rho > 0$ depending on a_1 and M (of (4)) such that*

$$\overline{\lim}_{n\to\infty} \|\phi^n(v)\| \le \rho$$

for all $v \in I\!R^N$, $h \in (0, h_0]$, $N \ge N_0$.

Easy calculation shows that

Lemma 1. *The eigenvalues of A are* $-\frac{4}{d^2} \sin^2 \frac{m\pi d}{2}$, $m = 1, \ldots, N$, *and the corresponding eigenvectors are*

$$(\sin m\pi d, \sin 2m\pi d, \ldots, \sin Nm\pi d) .$$

Proof. [of Theorem 1.] Substitute

$$v^{n+1} = \frac{v^{n+1} - v^n}{2} + \frac{v^{n+1} + v^n}{2}$$

into the left-hand side of

$$\left\langle \Delta v^{n+1}, v^{n+1} \right\rangle = \left\langle A v^{n+1} + \underline{f}(v^n) \ , \ v^{n+1} \right\rangle .$$

We obtain

$$\frac{\|v^{n+1}\|^2 - \|v^n\|^2}{2h} + \frac{\|v^{n+1} - v^n\|^2}{2h} = \left\langle A v^{n+1}, v^{n+1} \right\rangle + \left\langle \underline{f}(v^n), v^{n+1} \right\rangle . \tag{9}$$

Since $|f(u_1) - f(u_2)| \le M|u_1 - u_2|$ by (4), it follows that

$$\begin{aligned} \left\langle \underline{f}(v^n), v^{n+1} \right\rangle &= \left\langle \underline{f}(v^{n+1}), v^{n+1} \right\rangle + \left\langle \underline{f}(v^n) - \underline{f}(v^{n+1}), v^{n+1} \right\rangle \\ &\le \left\langle \underline{f}(v^{n+1}), v^{n+1} \right\rangle + M\|v^{n+1} - v^n\| \, \|v^{n+1}\|. \end{aligned} \tag{10}$$

Evidently for any $a > 0$ we have

$$\|v^{n+1} - v^n\| \cdot \|v^{n+1}\| \le \tfrac{a}{2} \|v^{n+1}\|^2 + \tfrac{1}{2a} \|v^{n+1} - v^n\|^2. \tag{11}$$

Fix $b > 0$ such that $a_1 + 2b < \pi^2$. Since A is symmetric, Lemma 1 implies for all $v \in I\!R^N$

$$\langle Av, v\rangle \le -\tfrac{4}{d^2}\sin^2\tfrac{\pi d}{2}\|v\|^2 \;\le\; -(a_1+2b)\|v\|^2$$

the latter for d small enough, i.e., $N \ge N_0$, say. Hence for $N \ge N_0$ and $v \in I\!R^N$

$$\langle Av\,,v\rangle + \left\langle \underline{f}(v)\,,v\right\rangle \;\le a_0 - 2b\|v\|^2\,. \tag{12}$$

Take $a \in (0, 2b/M)$, then we obtain from (10)-(12) that the right-hand side of (9) does not exceed

$$a_0 - b\|v^{n+1}\|^2 + \tfrac{M}{2a}\|v^{n+1} - v^n\|^2\,. \tag{13}$$

Now, if we take $h_0 = \frac{a}{M}$, then (9),(13) imply

$$\tfrac{1}{2h}[\|v^{n+1}\|^2 - \|v^n\|^2] \le a_0 - b\;\|v^{n+1}\|^2 \tag{14}$$

for $n \ge 0$, $N \ge N_0$, $h \in (0, h_0]$. Hence

$$\|v^{n+1}\|^2(1+2bh) \le \|v^n\|^2 + 2a_0h\,,$$

so that

$$\|v^n\|^2 \le (1+2bh)^{-n}\|v^0\|^2 + 2a_0h\sum_{m=1}^{n}(1+2bh)^{-m}$$

resulting in

$$\overline{\lim_{n\to\infty}}\|v^n\|^2 \le 2a_0h\sum_{m=1}^{\infty}(1+2bh)^{-m} = \frac{a_0}{b}\,.$$

Choosing now $\rho^2 = \frac{a_0}{b}$ completes the proof. $\square$

So, condition (8) guarantees the existence of a global attractor $I_{h,N}$ of $\phi_{h,N}$. We show now that the Hausdorff dimensions of these attractors are bounded uniformly in h, N.

Theorem 2. *For any $h \in (0, \frac{1}{M})$, $N \ge 1$, and any compact invariant set I of $\phi_{h,N}$ we have*

$$\dim_H I \le \kappa\,, \tag{15}$$

where κ is the smallest integer $\ge \sqrt{2M}$.

Proof. Recall that the singular values of a matrix A are the square roots of the eigenvalues of the matrix A^TA. It is well-known that if $N \times N$ matrices A_1, A_2, A_1A_2 have singular values $\sigma_1 \ge \cdots \ge \sigma_N$, $\tau_1 \ge \cdots \ge \tau_N$, $s_1 \ge \cdots \ge s_N$, respectively, then for $m = 1, \ldots, N$

$$s_1 \ldots s_m \le \sigma_1 \ldots \sigma_m \tau_1 \ldots \tau_m\,.$$

Let

$$\mu_1 \ge \cdots \ge \mu_N$$

be the singular values of $D\phi(v)$, the derivative of $\phi = \phi_{h,N}$, at $v \in I\!R^N$. We have

$$D\phi(v) = J^{-1}\,\mathrm{diag}(1 + hf'(v_1), \dots, 1 + hf'(v_N))\,.$$

As $J^{-1} = (E - hA)^{-1}$ is symmetric and nonnegative definite (see: Oliva et al., (1994)), its singular values coincide with its eigenvalues (see Lemma 1)

$$\left(1 + \tfrac{4h}{d^2}\sin^2\tfrac{m\pi d}{2}\right)^{-1}\,.$$

Evidently, the singular values of the matrix

$$\mathrm{diag}(1 + hf'(v_1), \dots, 1 + hf'(v_N))$$

do not exceed $1 + hM$. Hence for $\kappa = 1, \dots, N$

$$P(\kappa) = \mu_1 \dots \mu_\kappa \le \prod_{m=1}^{\kappa} \frac{1 + Mh}{1 + \frac{4h}{d^2}\sin^2\frac{m\pi d}{2}}\,.$$

It is shown in (Douady and Oesterlé, (1980)) that if for some κ

$$P(\kappa) < 1, \tag{16}$$

then for any compact invariant set I of ϕ we have $\dim_H I \le \kappa$. So, to prove Theorem 2 it remains to show that (16) holds, if $\kappa \ge \sqrt{2M}$.

Since $\sin t \ge \frac{2t}{\pi}$ for $t \in [0, \frac{\pi}{2}]$, we have

$$P(\kappa) \le \prod_{m=1}^{\kappa} \frac{1 + Mh}{1 + 4hm^2} = e^{\sum_{m=1}^{\kappa}[\log(1 + Mh) - \log(1 + 4hm^2)]}\,.$$

Using: $\log(1 + x) \ge \frac{x}{c}\log(1 + c)$ for $x \in [0, c]$ we get

$$\sum_{m=1}^{\kappa} \log(1 + 4hm^2) \ge \sum_{m=1}^{\kappa} \frac{m^2}{\kappa^2}\log(1 + 4h\kappa^2) > \frac{\kappa}{3}\log(1 + 4h\kappa^2)\,.$$

Hence $P(\kappa) < 1$, if

$$1 + 4h\kappa^2 > (1 + Mh)^3 = 1 + Mh(3 + 3Mh + M^2h^2)\,,$$

which is satisfied for $\kappa \ge \sqrt{2M}$ by (4) $\square$

For two functions $f, g \in C^p(I\!R)$, $p \ge 1$, and for a set $K \subset I\!R$ define

$$\rho_K^p(f, g) = \sum_{r=0}^{p} \sup_{x \in K} \left|\tfrac{\partial^r f}{\partial x^r} - \tfrac{\partial^r g}{\partial x^r}\right|.$$

We consider two standard topologies on $C^p(I\!R)$:

w : the topology of uniform C^p-convergence on compact subsets of $I\!R$. The base of neighborhoods of $f \in C^p(I\!R)$ in this topology consists of sets

$$\left\{g \,\middle|\, \rho_K^p(f, g) < \epsilon\right\}$$

for compact sets $K \subset I\!R$ and positive numbers ϵ. The space $C^p(I\!R)$ with this topology is denoted $\mathcal{F}_w^p$.

s : the strong C^p Whitney topology. The base of neighborhoods of f in this topology is defined as follows. Let $K = \{K_n\}$ be a countable family of compact subsets of $\mathbb{R}$ with

$$\bigcup_n K_n = \mathbb{R}\,,$$

and let $\epsilon = \{\epsilon_n\}$ be a sequence of positive numbers. Then the base consists of sets

$$\left\{g \,\middle|\, \rho^p_{K_n}(f,g) < \epsilon_n\right\}.$$

The space $C^p(\mathbb{R})$ with this topology is denoted $\mathcal{F}^p_s$.

Let $\mathcal{F}^p$ be one of the spaces $\mathcal{F}^p_w$, $\mathcal{F}^p_s$.

As usual, a subset of a topological space X is called *residual* in X if it contains a countable intersection of open dense sets in X. We say that a property P holds for *generic* element of X if the set

$$\left\{x \in X \,\middle|\, x \text{ satisfies } P\right\}$$

is residual.

Fix a compact set $K \subset \mathbb{R}^N$ and $\phi = \phi_{h,N}$.

Theorem 3. *For $p \geq 1$ the set*

$$\mathcal{H}^p(K) = \left\{f \in \mathcal{F}^p \,\middle|\, \text{fixed points of } \phi \text{ in } K \text{ are hyperbolic}\right\}$$

is residual in $\mathcal{F}^p$.

Proof. Fix $p \geq 1$ and $L > 0$ such that

$$K \subset \left\{v \,\middle|\, -L < v_m < L\,,\ m = 1, \dots, N\right\}.$$

As usually we say that a fixed point v of ϕ is *simple* if

$$\det(D\phi(v) - E) \neq 0\,. \tag{17}$$

Define

$$\mathcal{H} = \left\{f \in \mathcal{F}^p \,\middle|\, \text{ all fixed points of } \phi \text{ in } K \text{ are simple }\right\}.$$

We claim first that $\mathcal{H}$ is a residual subset of $\mathcal{F}^p$. A point v is a fixed point of ϕ if and only if

$$\Phi(v) = Av + \underline{f}(v) = 0\,.$$

For a map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 we say that a point x is *critical* if

$$\operatorname{rank} D\Psi(x) < n\,.$$

We denote by $S(\Psi)$ the set of critical points of Ψ. As

$$\Phi(v) = \tfrac{1}{h} J(\phi(v) - v)\,, \quad D\Phi(v) = \tfrac{1}{h} J(D\phi(v) - E)\,,$$

and $\det J \neq 0$, for a fixed point v of ϕ (17) is equivalent to $v \notin S(\Phi)$.

First consider the set

$$\mathbf{R}_0^N = \left\{ v \in I\!R^N \,\middle|\, v_i \neq v_j \text{ for } i \neq j \right\}$$

and $K_0 = \mathbf{R}_0^N \cap K$. The case of $\mathbf{R}_0^N$ will be a particular case of $\mathbf{R}_\Lambda^N$ (see below), but we treat it separately to clarify the main idea.

Let $\pi = (\pi_1, \ldots, \pi_N)$ be a permutation of $\{1, \ldots, N\}$. Take $l > 0$, $l_1, \ldots . l_N \in I\!R$ such that

$$-L < l_1 - 2l < l_1 + 2l < l_2 - 2l < \cdots < l_N + 2l < L\,, \tag{18}$$

denote $\lambda = (\, l\,, l_1, \ldots, l_N)$. Set

$$R_{0,\pi,\lambda} = \left\{ v \in \mathbf{R}_0^N \,\middle|\, l_i - l \leq v_{\pi_i} \leq l_i + l\,,\ i = 1, \ldots, N \right\}\,,$$

and

$$\mathcal{H}_{0,\pi,\lambda} = \left\{ f \in \mathcal{F}^p \,\middle|\, \text{ fixed points of } \phi \text{ in } R_{0,\pi,\lambda} \text{ are simple} \right\}\,.$$

Simple fixed points are isolated, hence there is only a finite number of them in a compact set. This evidently implies that $\mathcal{H}_{0,\pi,\lambda}$ is open.

Let us prove that $\mathcal{H}_{0,\pi,\lambda}$ is dense. By Sard's Theorem (see: Hirsch, (1976))

$$\operatorname{mes} \Phi(S(\Phi)) = 0$$

(here mes is the Lebesgue measure in $I\!R^N$). Therefore given $\epsilon > 0$ there exists $a \in I\!R^N$, $\|a\| < \epsilon$, such that

$$a \notin \Phi(S(\Phi))\,. \tag{19}$$

Consider $\tilde{f} \in \mathcal{F}^p$ such that

$$\tilde{f}(u) = f(u) - a_{\pi_m}$$

for

$$l_m - l \leq u \leq l_m + l\,, \quad m = 1, \ldots, N\,.$$

(Here a_{π_m} is the π_m–component of $a \in I\!R^N$.) Since (18) holds, l is fixed, and ϵ is arbitrary, $\tilde{f}$ can be found in an arbitrary $\mathcal{F}^p$-neighborhood of f. Set

$$\tilde{\Phi}(v) = Av + \underline{\tilde{f}}(v)\,. \tag{20}$$

Evidently, for $v \in R_{0,\pi,\lambda}$

$$\tilde{\Phi}(v) = \Phi(v) - a\,, \quad D\tilde{\Phi}(v) = D\Phi(v)\,. \tag{21}$$

Hence, if $v \in R_{0,\pi,\lambda}$ and $\tilde{\Phi}(v) = 0$, then $\Phi(v) = a$. It follows from (19),(21) that v is a simple fixed point of

$$\tilde{\phi}(v) = J^{-1}(v + h\underline{\tilde{f}}(v))\,.$$

Hence, $\mathcal{H}_{0,\pi,\lambda}$ is dense. As there exists a countable family of sets $\mathcal{H}_{0,\pi,\lambda}$ such that their union contains K_0, the set

$$\mathcal{H}_0 = \cap \mathcal{H}_{0,\pi,\lambda}$$

(we take the intersection corresponding to the family) is residual in $\mathcal{F}^p$, and for $f \in \mathcal{H}_0$ all fixed points of ϕ in K_0 are simple.

Now we consider a decomposition of $\{1, \dots, N\}$ into disjoint subsets $\Lambda_1, \dots, \Lambda_m$ numbered according to the lexicographic order, that is

$$m_1 < m_2 \quad \text{iff} \quad \min\{i \in \Lambda_{m_1}\} < \min\{i \in \Lambda_{m_2}\}\,.$$

Let $\Lambda = \{\Lambda_1, \dots, \Lambda_m\}$. Define the set

$$\mathbf{R}^N_\Lambda = \left\{ v \in I\!R^N \,\middle|\, v_i = v_j \text{ iff there exists } k \text{ with } i, j \in \Lambda_k \right\}\,.$$

The set $\mathbf{R}^N_0$ defined earlier corresponds to

$$\Lambda = \{\{1\}, \dots, \{N\}\}\,.$$

Let $\pi = (\pi_1, \dots, \pi_m)$ be a permutation of $\{1, \dots, m\}$.
Take $l > 0$, $l_1, \dots .l_m \in I\!R$ such that

$$-L < l_1 - 2l < l_1 + 2l < l_2 - 2l < \cdots < l_m + 2l < L\,,$$

let $\lambda = (\,l\,, l_1, \dots, l_m)$,

$$R_{\Lambda,\pi,\lambda} = \left\{ v \in \mathbf{R}^N_\Lambda \,\middle|\, l_i - l \le v_j \le l_i + l \;\text{ if }\; j \in \Lambda_{\pi_i}\,,\; i = 1, \dots, m \right\}\,,$$

and

$$\mathcal{H}_{\Lambda,\pi,\lambda} = \left\{ f \in \mathcal{F}^p \,\middle|\, \text{ fixed points of } \phi \text{ in } R_{\Lambda,\pi,\lambda} \text{ are simple} \right\}\,.$$

Evidently, any $\mathcal{H}_{\Lambda,\pi,\lambda}$ is open in $\mathcal{F}^p$. Let us show that $\mathcal{H}_{\Lambda,\pi,\lambda}$ is dense. Denote by p_Λ a projection of $I\!R^N$ onto the linear subspace

$$\mathcal{R}_\Lambda = \overline{\mathbf{R}^N_\Lambda} = \left\{ v \in I\!R^N \,\middle|\, v_i = v_j \text{ if there exists } k \text{ with } i, j \in \Lambda_k \right\}\,.$$

Note that for $v \in \mathcal{R}_\Lambda$ we have $\underline{f}(v) \in \mathcal{R}_\Lambda$.

Consider the map

$$\Phi_\Lambda = p_\Lambda \Phi\Big|_{\mathcal{R}_\Lambda}\,,$$

which evidently is in C^1.

It follows from the Sard's Theorem (we take into account that Φ_Λ acts from $\mathcal{R}_\Lambda$ into $\mathcal{R}_\Lambda$) that

$$\text{mes}_\Lambda\, \Phi_\Lambda(S(\Phi_\Lambda)) = 0$$

(mes_Λ is the Lebesgue measure in $\mathcal{R}_\Lambda$). Given $\epsilon > 0$ there exists $a \in \mathcal{R}_\Lambda$, $\|a\| < \epsilon$ with

$$a \notin \Phi_\Lambda(S(\Phi_\Lambda))\,.$$

Consider $\tilde{f} \in \mathcal{F}^p$ such that

$$\tilde{f}(u) = f(u) - a_j$$

for $u \in [l_i - l, l_i + l]$ if $j \in \Lambda_{\pi_i}$, $i = 1, \dots, m$. Then for $v \in \mathcal{R}_\Lambda$

$$\underline{\tilde{f}}(v) = \underline{f}(v) - a\,.$$

The same reasons as in the case of $R_{0,\pi,\lambda}$ show that for $v \in R_{\Lambda,\pi,\lambda}$ with $\tilde{\Phi}_\Lambda(v) = 0$ we have

$$\operatorname{rank} D\tilde{\Phi}_\Lambda(v) = \dim \mathcal{R}_\Lambda , \tag{22}$$

where

$$\tilde{\Phi}_\Lambda = p_\Lambda(Av + \underline{\tilde{f}}(v)) .$$

Therefore the corresponding $\tilde{\phi}$ has a finite number of fixed points in $R_{\Lambda,\pi,\lambda}$, and for these fixed points (22) holds. Evidently, $\tilde{f}$ can be found in an arbitrary $\mathcal{F}^p$–neighborhood of f .

Now we show that these fixed points can be made simple. For simplicity of notation we assume that ϕ itself has a finite number of fixed points, $w^1, \ldots, w^s$, in $R_{\Lambda,\pi,\lambda}$, and (22) holds for Φ instead of $\tilde{\Phi}$. It follows that given a small neighborhood V_i of w^i (in $\mathcal{R}_\Lambda$) we can find a neighborhood W_i of f in $\mathcal{F}^p$ such that if $\tilde{f} \in W_i$ and

$$\tilde{f}(w^i_m) = f(w^i_m), \; m = 1, \ldots, N , \tag{23}$$

then w^i is the unique fixed point of $\tilde{\phi}$ in V_i .

Take $\tilde{f}$ such that (23) holds for $w^1, \ldots, w^s$, and

$$\tilde{f}'(w^k_m) = f'(w^k_m) + \mu , \; k = 1, \ldots, s , \; m = 1, \ldots, N , \tag{24}$$

where $\mu \in I\!R$. For w^k let

$$p^k_N(\mu) = \det(D\tilde{\phi}(w^k) - E) = \det[J^{-1}(E + h \operatorname{diag}(f'(w^k_1) + \mu, \ldots, f'(w^k_N) + \mu)) - E] .$$

Easy computation shows that

$$p^k_N(\mu) = \frac{h^N}{\det J} \mu^N + \text{ lower degree terms.}$$

Take small disjoint neighborhoods V_k of points w^k and find corresponding neighborhoods W_k of f in $\mathcal{F}^p$. Then there exists a neighborhood W_0 of f in $\mathcal{F}^p$ such that for $\tilde{f} \in W_0$ the corresponding $\tilde{\phi}$ has no fixed points in

$$R_{\Lambda,\pi,\lambda} \setminus (V_1 \cup \cdots \cup V_s) .$$

So, if we take $\tilde{f} \in W_0$ such that (23), (24) hold, then the set of fixed points of $\tilde{\phi}$ in $R_{\Lambda,\pi,\lambda}$ coincides with $\{w^1, \ldots, w^s\}$.

Let μ_0 be the least positive root of the polynomials $p^1_N, \ldots, p^s_N$. As the set of coordinates of $w^1, \ldots, w^s$ is finite, we can find $\mu \in (0, \mu_0)$, and a corresponding $\tilde{f}$ which satisfies (23),(24), in an arbitrary neighborhood of f in $\mathcal{F}^p$. For this $\tilde{f}$ all fixed points of $\tilde{\phi}$ in $R_{\Lambda,\pi,\lambda}$ are simple. Hence, $\mathcal{H}_{\Lambda,\pi,\lambda}$ is dense.

The given compact set K is a subset of a countable union of sets $R_{\Lambda,\pi,\lambda}$. Note that if $\Lambda = \{\Lambda_1\}$, that is if

$$\mathcal{R}_\Lambda = \left\{ v \,\middle|\, v_1 = \cdots = v_N \right\} ,$$

then $\mathcal{R}_\Lambda \cap K$ belongs to one set $R_{\Lambda,\pi,\lambda}$.

Hence, the set $\mathcal{H} = \cap \mathcal{H}_{\Lambda,\pi,\lambda}$ is residual in $\mathcal{F}^p$. As $\mathcal{F}^p$ is a complete metric space (see: Hirsch, (1976)), $\mathcal{H}$ is dense in $\mathcal{F}^p$.

Evidently, the set $\mathcal{H}^p(K)$ is open. Let us prove that $\mathcal{H}^p(K)$ is dense. Take $f \in \mathcal{F}^p$. As $\mathcal{H}$ is dense, we may take $f \in \mathcal{H}$.

Let $u_1, \dots, u_s$ be all the coordinates of fixed points of ϕ in K. It was shown earlier that for $f \in \mathcal{H}$ there exists a neighborhood W of f in $\mathcal{F}^p$ such that if $\tilde{f} \in W$, and

$$\tilde{f}(u_i) = f(u_i)\,, \; i = 1, \dots, s\,, \tag{25}$$

then the set of fixed points of $\tilde{\phi}$ in K coincides with that of ϕ.

Fix $\epsilon > 0$ and take $\tilde{f}$, which satisfies (25) and such that

$$\tilde{f}'(u_i) = \tfrac{1}{1+\epsilon}(f'(u_i) - \tfrac{\epsilon}{h})\,,$$

then

$$1 + h\tilde{f}'(u_i) = \tfrac{1}{1+\epsilon}\,(\,1 + hf'(u_i)\,)\,, \; i = 1, \dots, s\,.$$

Hence, for any fixed point $v \in K$

$$D\tilde{\phi}(v) = \tfrac{1}{1+\epsilon}\,D\phi(v)\,.$$

Therefore, the eigenvalues μ_j and λ_j of $D\tilde{\phi}(v)$ and $D\phi(v)$, respectively, are related by

$$\mu_j = \tfrac{1}{1+\epsilon}\lambda_j\,.$$

Evidently, we can find ϵ so small that $\tilde{f}$ is in a given neighborhood of f in $\mathcal{F}^p$, and for all μ_j we have

$$|\mu_j| \neq 1\,.$$

Hence, fixed points of $\tilde{\phi}$ in K are hyperbolic. This completes the proof. $\square$

Remark 2. *An analogous statement for the semigroup $S(t)$ (for C^p–generic f, $p \geq 2$, all equilibrium solutions of (1) are hyperbolic) was obtained in (Brunovsky and Chow, (1984)) using quite different ideas, based on detailed investigation of the corresponding ODE:* $\quad u'' + f(u) = 0\,.$

By definition, a diffeomorphism is a Morse-Smale diffeomorphism (see e.g. Pilyugin, (1992)) if:

1) its nonwandering set Ω is finite and consists of hyperbolic periodic points;
2) the stable and unstable manifolds of points of Ω are transversal.

In a computer the time step h takes rational values, so, it is reasonable to consider a fixed countable set H of h–values. Then we obtain from Theorems 0,3 the following.

Theorem 4. *For $p \geq 1$ there exists a residual subset F^p of $\mathcal{F}^p_w$ (or $\mathcal{F}^p_s$) such that if $f \in F^p$, then $\phi = \phi_{h,N}$ is a Morse-Smale diffeomorphism for any $(h, N) \in H \times \mathbb{N}$.*

3. Pseudotrajectories of $\phi_{h,N}$

Lemma 2. *There exists* $l = l(h, N)$ *such that any sequence* w^n, *which satisfies* (5), *is an* $l\delta$*–trajectory of* $\phi = \phi_{h,N}$.

Proof. Write (5) as

$$\Delta w^{n+1} = Aw^{n+1} + \underline{f}(w^n) + z^n ,$$

where $\|z^n\| < \delta$. As

$$w^{n+1} = J^{-1}(w^n + h\underline{f}(w^n)) + hJ^{-1}z^n = \phi(w^n) + hJ^{-1}z^n ,$$

we see that our lemma is proved with $l = h\|J^{-1}\|$. □

Now we apply results of (Pilyugin, (1994a)). Let I be an attractor of ϕ. There exists an open set V such that

$$I \subset V \subset \overline{V} \subset D(I) , \quad \phi\left(\overline{V}\right) \subset V .$$

We define a parametrizing set F for $D(I) \setminus I$ by

$$F = \overline{V} \setminus \phi\left(\overline{V}\right) .$$

We say that $\Xi = \left\{\xi(p) \,\middle|\, p \in I\!R^N\right\}$ is a *complete family of* Δ*–trajectories of* ϕ (denote $CF(\Delta, \phi)$) if for any $p \in I\!R^N$

$$\xi(p) = \left\{\xi^k(p)\right\}_{k\geq 0}$$

is a Δ–trajectory with $\xi^0(p) = p$. Fix a natural T and for Ξ — a $CF(\Delta, \phi)$ — set

$$\Xi(T, F) = \left\{\xi^T(p) \,\middle|\, p \in F\right\} , \quad \Xi(T, \infty, F) = \bigcup_{n\geq T} \Xi(n, F) .$$

The Hausdorff distance R is defined as usually: for A, B compact in $I\!R^N$ let

$$r(A, B) = \max_{x\in A} \min_{y\in B} \|x - y\| , \quad R(A, B) = \max(r(A, B), r(B, A)) .$$

It follows from Theorems 1 and 2 of (Pilyugin, (1994a)), that if I is an attractor of ϕ with $\operatorname{Int} I = \emptyset$, and F is a paremetrizing set for I, then given $\epsilon > 0$ there exist $\Delta, T > 0$ such that if Ξ is a $CF(\Delta, \phi)$, then

$$R(I, \Xi(T, F)) < \epsilon \quad \text{and} \quad R(I, \Xi(T, \infty, F)) < \epsilon . \tag{26}$$

If now $N > \kappa$ (see Theorem 2), then $\operatorname{Int} I = \emptyset$ and Lemma 2 implies the following:

Theorem 5. *Let* $N > \kappa$. *Assume that* I *is an attractor of* ϕ *and let* F *be a parametrizing set for* I. *Given* $\epsilon > 0$ *there exist* $T, \delta > 0$ *such that for any family of sequences* $\{w^n\}, w^0 \in F$, *which satisfies* (5), (26) *holds, where*

$$\Xi = \left\{\{w^n\}_{n\geq 0} \,\middle|\, w^0 \in F\right\} .$$

Evidently the set

$$\mathcal{B} = \bigcup_{a_1 < \pi^2} \mathcal{B}_{a_0,a_1}$$

(see Theorem 1) is open in $\mathcal{F}_s^p$. By Theorem 4 there is a residual (in the topology of $\mathcal{F}_s^p$) subset $\mathcal{B}_0$ of $\mathcal{B}$ such that for $f \in \mathcal{B}_0$ $\phi = \phi_{h,N}$ is a Morse-Smale diffeomorphism for any $(h, N) \in H \times \mathbb{N}$.

In this case we can establish some additional properties. Let $S_a(A)$ be the open a-neighborhood of $A \subset \mathbb{R}^N$. We say that a set $G \subset \mathbb{R}^N$ absorbs sequences $\{w^n\}$ which satisfy (5) if the inclusion $w^n \in G$ implies $w^{n+1} \in G$ (and hence $w^{n+m} \in G$ for all $m > 0$).

Theorem 6. *Assume that $f \in \mathcal{B}_0$, let $I = I_{h,N}$ be an attractor of $\phi = \phi_{h,N}$. Let F be a parametrizing set for $D(I) \setminus I$. There exist numbers ϵ_0, m, L, $\delta_0 > 0$, $Q \in (0, 1)$ such that for any family of sequences $\{w^n\}_{n \geq 0}$, $w^0 \in F$, which satisfy (5), the following statements hold:*

1) *if $\delta \in (0, \delta_0)$, then for a given sequence $\{w^n\}$ there is a $v \in \mathbb{R}^N$ such that*

$$\|\phi^n(v) - w^n\| \leq L\delta\,, \quad n \geq 0\,;$$

2) *for any $\epsilon \in (0, \epsilon_0)$*
   2.1) *there exists a set U_ϵ which absorbs sequences $\{w^n\}$ with*

$$0 < \delta < \min(\delta_0\,,\, \epsilon^m / L)\,,$$

and

$$S_{\epsilon L^{-1}}(I) \subset U_\epsilon \subset \overline{S_\epsilon(I)}\,;$$

   2.2) *$w^n \in U_\epsilon$ for*

$$n \geq m \log_Q \epsilon - \log_Q L\,.$$

Proof. If ϕ is a Morse-Smale diffeomorphism, then ϕ satisfies the strong transversality condition (see: Pilyugin, (1992)) on every invariant set. The same proof as the one given in Appendix A of the book (Pilyugin, (1994b)) shows that there is a neighborhood U of I and numbers $L_0, \Delta_0 > 0$ such that if $\delta < \Delta_0$ and $w^n \in U$ for $n \geq n_0$ then there exists v^0 with the property:

$$\|\phi^m(v^0) - w^{n_0+m}\| \leq L_0\delta \quad \text{for} \quad m \geq 0\,.$$

By Lemma 2 in (Pilyugin, (1994a)) there exists $\delta_0 < \Delta_0$ and $n_0 = n_0(F, U)$ such that if $\delta \in (0, \delta_0)$, then $w^n \in U$ for $n \geq n_0$. Evidently there exists $L_1 > 0$ such that

$$\|\phi^{-n}(v^0) - w^{n_0-n}\| \leq L_1\delta \quad \text{for} \quad n = 0, \ldots, n_0\,.$$

This implies the first statement. The second one follows directly from Theorem 5 in (Pilyugin, (1994a)). □

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