LIMIT SHADOWING PROPERTY

Timo Eirola *, Olavi Nevanlinna *, and Sergei Yu. Pilyugin #

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* Institute of Mathematics, Helsinki University of Technology, Finland
# Faculty of Mathematics and Mechanics, University of St. Petersburg, Russia

Abstract. In this paper pseudoorbits of discrete dynamical systems are considered such that the one–step errors of the orbits tend to zero with increasing indices. First it is shown that close to hyperbolic sets such orbits are shadowed by true trajectories of the system with shadowing errors also tending to zero. Then the rates of convergence are studied via considering pseudoorbits such that the error sequences belong to certain (weighted) l_p–spaces and showing that the corresponding shadowing errors are there, too. Under certain conditions on the weights we establish weighted shadowing near nonhyperbolic sets.

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1 Introduction

In simulation of dynamical systems improved accuracy usually has to be paid by increasing computation power or time. This suggests the natural simulation strategy to use low accuracy until "interesting" things are observed and reduce errors during computation in the hope that a true trajectory is approached and real dynamics of it is captured. This is an example of a situation where the one–step errors of pseudoorbits approximating trajectories of a dynamical system tend to zero. In this paper we look for trajectories that shadow such pseudoorbits and especially such for which the shadowing errors also tend to zero.

Let ϕ be a dynamical system on a metric space (X, d). It is said that a sequence $\{x_k\}_{k\in K}$, $K \subset \mathbf{Z}$, is ε–*shadowed* (or ε–traced) by the trajectory through $x \in X$ if

$$d(\phi^k(x), x_k) < \varepsilon$$

for all $k \in K$. The system has the *pseudoorbit tracing property* (POTP) if for given $\varepsilon > 0$

there exists $\delta > 0$ such that any δ–*trajectory*[1], i.e., a sequence $\{x_k\}_{k\in\mathbf{Z}}$ with

$$d(x_{k+1}, \phi(x_k)) < \delta$$

is ε–*shadowed* (or ε–traced) by a trajectory of ϕ.

A survey of shadowing results can be found in the book [8]. Often pseudotrajectories are obtained as results of numerical studies of dynamical systems. In this context POTP means that numerically found trajectories with uniformly small errors are close to real trajectories.

In this paper we study a shadowing property close to the one described above. We say that ϕ has the *limit shadowing property* if for any sequence $\{x_k\}_{k\geq 0}$ with

$$d(x_{k+1}, \phi(x_k)) \underset{k\to\infty}{\to} 0$$

there is a point x such that

$$d(\phi^k(x), x_k) \underset{k\to\infty}{\to} 0 \ .$$

And similarly for two–sided sequences $\{x_k\}_{k=-\infty}^{\infty}$.

From the numerical point of view this property of a dynamical system ϕ means the following: if we apply a numerical method that approximates ϕ with "improving accuracy" so that one step errors tend to zero as time goes to infinity, then the numerically obtained trajectories tend to real ones. Such situations arise for example when we are not so interested in the initial (transient) behaviour of trajectories but want to get to areas where "interesting things" happen (e.g. attractors) and then improve accuracy.

In section 2 we study limit shadowing of a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$. It is shown that in a neighbourhood of a hyperbolic set ϕ has the limit shadowing property (Theorem 2.1). Then in section 3 we study the rate of convergence of the sequence $\left\{|\phi^k(x) - x_k|\right\}_{k\geq 0}$ to zero. We prove that in a neighbourhood of a hyperbolic set Λ the following holds: if

$$\Big(\sum_{k\in\mathbf{Z}} |x_{k+1} - \phi(x_k)|^p\Big)^{1/p} < \delta \tag{1.1}$$

for some $p \geq 1$ and small enough δ, then there exists an x such that

$$\Big(\sum_{k\in\mathbf{Z}} |x_k - \phi^k(x)|^p\Big)^{1/p} < L\delta \tag{1.2}$$

(Theorem 3.1). We call this l_p–shadowing. Further this is extended to weighted norms

$$\|v\|_{r,p} = \Big(\sum_{k\in\mathbf{Z}} r_k |v_k|^p\Big)^{1/p}$$

under a condition on the weights r_k and the hyperbolicity characteristics of ϕ (Theorems 3.3, 3.4, and 3.6). Under some conditions on the weights we do not assume the invariant set to be hyperbolic. Finally, in section 4 we see what these results mean in an example of a hyperbolic periodic flow.

[1] also called δ–orbit or δ–pseudotrajectory or δ–pseudoorbit

2 Preliminaries & limit shadowing of positive pseudo–orbits

Let ϕ be a C^1–diffeomorphism of $\mathbb{R}^n$. Denote by $D\phi(x)$ its derivative. Assume $\Lambda \subset \mathbb{R}^n$ is a hyperbolic set of ϕ, that is

(a) Λ is compact and ϕ–invariant;

(b) there exist constants $C > 0$, $\lambda \in (0,1)$ and a family of linear subspaces E^s_x, E^u_x, $x \in \Lambda$, of $\mathbb{R}^n$ such that

(b1) $E^s_x \oplus E^u_x = \mathbb{R}^n\,, \quad x \in \Lambda\,;$

(b2) $D\phi(x)E^\sigma_x = E^\sigma_{\phi(x)}\,, \quad x \in \Lambda\,,\ \sigma = s, u\,;$

(b3)

$$
\begin{aligned}
&(2.3) \qquad |D\phi^m(x)v\,| \le C\lambda^m |v| \qquad \text{for} \quad v \in E^s_x\,,\ m \ge 0\\
&(2.4) \qquad |D\phi^{-m}(x)v\,| \le C\lambda^m |v| \qquad \text{for} \quad v \in E^u_x\,,\ m \ge 0\,.
\end{aligned}
$$

We can introduce the so–called Lyapunov (or adapted) metric d' in a neighbourhood U_0 of Λ (see: e.g. [11]). It has the following properties:

$$
(2.5) \qquad a|x-y| \le d'(x,y) \le b|x-y|
$$

for some a, $b > 0$, and with respect to the norm generated by d' we can take $C = 1$ in (2.3) and (2.4) with some $\lambda' \in (\lambda, 1)$ instead of λ. We work with d', λ' below preserving the previous notation, that is $|x-y|$ stands for $d'(x,y)$ and λ for λ'. It follows from inequalities (2.5) that all results of this section are valid in the case of the standard Euclidean metric.

The hyperbolic structure $\{E^s_x, E^u_x\}$ can be extended to a neighbourhood of Λ. More exactly:

Lemma 2.1 (See: [2]) *Fix $\mu \in (\lambda, 1)$. For any $\nu > 0$ there exists a neighbourhood $U_1 = U_1(\mu,\nu)$ of Λ and continuous families of linear subspaces $\widetilde{E}^s_x$, $\widetilde{E}^u_x$, $x \in U_1$, with the properties*

a) $\widetilde{E}^\sigma_x = E^\sigma_x\,, \quad x \in \Lambda\,,\ \sigma = s, u\,;$

b) $\widetilde{E}^s_x \oplus \widetilde{E}^u_x = \mathbb{R}^n\,, \quad x \in U_1\,;$

c) if P^s_x is the projection of $\mathbb{R}^n$ onto $\widetilde{E}^s_x$ along $\widetilde{E}^u_x$ and $P^u_x = I - P^s_x$, then for $x \in U_1$ with $\phi(x) \in U_1$, we have the following:

$$
(2.6) \qquad \text{if } w \in \widetilde{E}^s_x\,, \text{ then } \quad |P^s_{\phi(x)}D\phi(x)w| \le \mu|w|\,, \quad |P^u_{\phi(x)}D\phi(x)w| \le \tfrac{\nu}{2}|w|\,,
$$

$$
(2.7) \qquad \text{if } w \in \widetilde{E}^u_x\,, \text{ then } \quad |P^u_{\phi(x)}D\phi(x)w| \ge \tfrac{1}{\mu}|w|\,, \quad |P^s_{\phi(x)}D\phi(x)w| \le \tfrac{\nu}{2}|w|\,.
$$

For a sequence of points $\xi = \{x_k\}_{k \in \mathbf{Z}} \subset \mathbb{R}^n$ and for a point x set

$$d(\xi) = \{\phi(x_k) - x_{k+1}\}_{k \in \mathbf{Z}} \ , \quad d^+(\xi) = \{\phi(x_k) - x_{k+1}\}_{k \geq 0} \ ,$$

$$\Delta(x, \xi) = \left\{\phi^k(x) - x_k\right\}_{k \in \mathbf{Z}} \ , \quad \Delta^+(x, \xi) = \left\{\phi^k(x) - x_k\right\}_{k \geq 0} \ .$$

First we recall some definitions and preliminary results about shadowing in a neighbourhood of hyperbolic sets. We say that ξ is a δ–trajectory of ϕ if

$$|d(\xi)_k| < \delta \ , \quad k \in \mathbf{Z} \, . \tag{2.8}$$

Similarly, we call $\xi = \{x_k\}_{k \geq 0}$ a positive δ–trajectory, if the inequality of (2.8) is satisfied for $k \geq 0$.

Lemma 2.2 (A weak variant of Lemma A.1 of [8]). *There exists a neighbourhood U_2 of Λ and positive numbers L_2 and δ_2 such that if $\xi \subset U_2$ is a δ–trajectory with $\delta < \delta_2$, then there exists a point x^* such that*

$$|\Delta(x^*, \xi)_k| \leq L_2 \delta \ , \quad k \in \mathbf{Z} \ .$$

And similarly for positive δ–trajectories.

Remark. The proof in [8] is based on ideas of Bowen [3]. One can prove Lemma 2.2 also using techniques of Theorem 3.1 below. This approach was developed by Anosov [2].

It follows from Lemma 2.1 and from the Perron theorem that trajectories of ϕ have stable and unstable manifolds while they remain in a neighbourhood of Λ (see [9] for details), so the following holds.

Lemma 2.3 *There exists a neighbourhood U_3 of Λ and $\delta_3 > 0$ such that if for two points x, y we have $\phi^k(x), \phi^k(y) \in U_3$ for $k \geq 0$ and*

$$|\phi^k(x) - \phi^k(y)| \leq \delta_3 \ , \quad k \geq 0 \ , \tag{2.9}$$

then

$$|\phi^k(x) - \phi^k(y)| \leq 2\left(\tfrac{1+\mu}{2}\right)^k |x - y| \ , \quad k \geq 0 \ . \tag{2.10}$$

Remark. Inequality (2.9) with small δ_3 means that x and y belong to the same local stable manifold, hence (2.10) holds.

The following theorem is a pure limit shadowing statement for positive pseudo–orbits. It says that if the one–step errors tend to zero, then there exists a true trajectory such that the shadowing errors tend to zero. For two–sided sequences see Prop. 3.2.

Theorem 2.1 *There exists a neighbourhood U of Λ such that if a sequence $\xi = \{x_k\}_{k \geq 0}$ belongs to U and if*

$$d^+(\xi)_k \underset{k \to \infty}{\to} 0 \ , \tag{2.11}$$

then there is a point x^* such that

$$\Delta^+(x^*, \xi)_k \underset{k \to \infty}{\to} 0 \ . \tag{2.12}$$

Remark. For a fixed point Λ this result was proved in [1] (Thm.13, chapter 9). One can prove our Thm. 2.1 by applying Proposition 11, chapter 10 in [1], but we prefer to give here a simple direct proof.

Proof. Take L_2, δ_2, δ_3, U_2, and U_3 from Lemmas 2.2 and 2.3 and set $\widetilde{U} = U_2 \cap U_3$. Take $\varepsilon_0 > 0$ and a neighbourhood U of Λ such that the open ε_0–neighbourhood of U is in $\widetilde{U}$. Let $\xi = \{x_k\}_{k\geq 0} \subset U$ be such that (2.11) holds. Take $\varepsilon > 0$ such that

$$\varepsilon < \min(L_2\delta_2, \tfrac{\delta_3}{2}, \varepsilon_0) \ . \tag{2.13}$$

For $j = 1, 2, \ldots$ find (by (2.11)) an index k_j such that $d^+(\xi)_k < \frac{\varepsilon}{L_2 j}$ for $k > k_j$. By Lemma 2.2 and (2.13) there exist points y_j such that

$$|\phi^k(y_j) - x_k| < \tfrac{\varepsilon}{j} \ , \quad k > k_j \ . \tag{2.14}$$

It follows from (2.13) that $\phi^k(y_j) \in \widetilde{U}$ for $k > k_j$. We claim that (2.12) holds for $x^* = y_1$. Indeed, it follows from (2.14) and the choice of ε that

$$|\phi^k(y_1) - \phi^k(y_j)| < 2\varepsilon < \delta_3 \ , \quad k > k_j \ .$$

Hence by Lemma 2.3

$$|\phi^k(y_1) - \phi^k(y_j)| < 4\varepsilon \left(\tfrac{1+\mu}{2}\right)^{k-k_j} \ , \quad k > k_j \ . \tag{2.15}$$

For each j take $l_j > 0$ such that

$$4\left(\tfrac{1+\mu}{2}\right)^{l_j-k_j} < \tfrac{1}{j} \ . \tag{2.16}$$

Then (2.14)-(2.16) imply

$$|x_k - \phi^k(y_1)| < \tfrac{2\varepsilon}{j} \ , \quad k \geq l_j \ ,$$

which completes the proof. □

3 l_p–shadowing

The following states the l_p–shadowing result. Here we consider two–sided sequences, i.e., indices running from $-\infty$ to ∞. The theorem says that if the sequence of the one–step errors is in l_p, then the same holds also for the sequence of the shadowing errors.

For $p \geq 1$ denote by l_p be the Banach space of sequences $v = \{v_k\}_{k\in\mathbf{Z}}$ with the norm

$$\|v\|_p = \Big(\sum_{k\in\mathbf{Z}} |v_k|^p\Big)^{1/p} \ .$$

Also denote $\|v\|_\infty = \sup_{k\in\mathbf{Z}} |v_k|$.

Note that in Theorems 3.1, 3.3, and in Proposition 3.2, Λ and ϕ are as in section 2.

Theorem 3.1 *There exists a neighbourhood U of Λ and numbers L, $\delta_0 > 0$ such that if for some $p \in [1, \infty)$ and for a sequence $\xi = \{x_k\}_{k \in \mathbf{Z}} \subset U$ we have $\|d(\xi)\|_p < \delta_0$, then there exists a unique x^* such that*

$$\|\Delta(x^*, \xi)\|_p \leq L \|d(\xi)\|_p \ . \tag{3.1}$$

Proof. We begin with an elementary remark. Let B be a Banach space represented as

$$B = B^s \oplus B^u \ . \tag{3.2}$$

Assume that for a linear operator $A: \ B \to B$ represented as

$$A = \begin{bmatrix} A^{ss} & A^{su} \\ A^{us} & A^{uu} \end{bmatrix}$$

according to (3.2), we have

$$\|A^{ss}\| \leq \mu_0 \ , \quad \|(A^{uu})^{-1}\| \leq \mu_0$$

for some $\mu_0 \in (0,1)$. Then there exist ν, $R > 0$ such that if

$$\|A^{su}\| \leq \nu \ , \quad \|A^{us}\| \leq \nu \ ,$$

then $A - I$ is invertible and $\left\|(A-I)^{-1}\right\| \leq R$. Fix $\mu \in (\lambda, 1)$, take $\mu_0 = \frac{1+\mu}{2}$, find corresponding ν, R. Apply Lemma 2.1 to find a neighbourhood $U_1 = U_1(\mu, \nu)$ of Λ. Take a neighbourhood U of Λ with compact closure such that $\overline{U} \cup \phi(\overline{U}) \subset U_1$. Denote

$$B_p(r) = \left\{ u \in l_p \mid \|u\|_p \leq r \right\} \ .$$

Given a sequence $\xi = \{x_k\}_{k \in \mathbf{Z}}$ let $\delta = \|d(\xi)\|_p$ and $\eta = \{y_k\}_{k \in \mathbf{Z}}$, where $y_k = x_k + v_k$. Then η is a trajectory of ϕ if and only if $v = \{v_k\}_{k \in \mathbf{Z}}$ satisfies

$$v_{k+1} = \phi(x_k + v_k) - x_{k+1} \quad , \qquad k \in \mathbf{Z} . \tag{3.3}$$

Obviously we have

$$\phi(x_k + v_k) - x_{k+1} = [\phi(x_k) - x_{k+1}] + D\phi(x_k) v_k + \psi_k(v_k) \ , \tag{3.4}$$

where $\psi_k(0) = 0$ and $\frac{\partial \psi_k}{\partial v_k}(0) = 0$.

Since $\overline{U}$ is compact, there exist constants C_1, $\delta_1 > 0$ and a function $\beta(s) \underset{s \to 0}{\to} 0$ such that

$$\|D\phi(x)\| \leq C_1 \ , \quad x \in U \ , \text{ and}$$

$$|\psi_k(v_k) - \psi_k(v'_k)| \leq \beta(\max(\|v\|_\infty, \|v'\|_\infty)) \, |v_k - v'_k| \quad \text{for} \quad \|v\|_\infty, \|v'\|_\infty \leq \delta_1 \ .$$

Since $\|v\|_\infty \leq \|v\|_p$, it follows from the inequalities

$$|\phi(x_k + v_k) - x_{k+1}| \leq |d(\xi)_k| + (C_1 + \beta(\delta_1)) \, |v_k|$$

that, for $\delta < \delta_1$, $F(v) = \widetilde{v}$, where

$$\widetilde{v}_k = \phi(x_{k-1} + v_{k-1}) - x_k \quad , \qquad k \in \mathbf{Z}$$

defines an operator from $B_p(\delta_1)$ to l_p. The set $\eta = \{x_k + v_k\}_{k\in\mathbf{Z}}$ is a trajectory of ϕ if and only if v is a fixed point of F. It follows from (3.4) that F is differentiable at 0 with $(DF(0)v)_k$ depending only on v_{k-1}.

Consider an extended hyperbolic structure $\{\widetilde{E}^s_x,\ \widetilde{E}^u_x\}$ on U given by Lemma 2.1. Below we write E^s_x, E^u_x instead of $\widetilde{E}^s_x$, $\widetilde{E}^u_x$. We can represent

$$l_p = l^s_p \oplus l^u_p \ ,$$

where

$$l^\sigma_p = \Big\{ v \in l_p \,|\, v_k \in E^\sigma_{x_k},\ k \in \mathbf{Z} \Big\} \ , \quad \sigma = s, u \, .$$

Take $\gamma \in (1,2)$ such that $\gamma\mu < (1+\mu)/2 = \mu_0$. Take the corresponding ν, R. Suppose that $U \subset U_1(\mu_0, \nu)$ (see Lemma 2.1). By continuity of the families E^s_x, E^u_x on the compact set $\overline{U}$, we can find $\delta_2 > 0$ with the property: if $x, y \in U$, $|x-y| < \delta_2$, then for $v^{\,\sigma} \in E^\sigma_x$, $\sigma = s, u$, the decomposition

$$v^{\,\sigma} = \widetilde{v}^{\,s} + \widetilde{v}^{\,u} \ , \quad \widetilde{v}^{\,\tau} \in E^\tau_y \, , \quad \text{satisfies} \quad |\widetilde{v}^{\,\sigma}| \le \gamma |v^{\,\sigma}| \, . \tag{3.5}$$

Assume that $\delta < \delta_2$. Let us show that for $w = w^s + w^u$, where $w^\sigma = P^\sigma_{x_k} w$, the decomposition of $\widetilde{w} = D\phi(x_k)w$:

$$\widetilde{w} = \widehat{w}^{ss} + \widehat{w}^{su} + \widehat{w}^{us} + \widehat{w}^{uu} \ ,$$

where $\widehat{w}^{\sigma\tau} = P^\sigma_{x_{k+1}} D\phi(x_k) w^\tau$, $\sigma, \tau \in \{s, u\}$ satisfies

$$\begin{array}{ll} |\widehat{w}^{ss}| \le \mu_0 |w^s| \ , & |\widehat{w}^{su}| \le \nu |w^u| \\ |\widehat{w}^{us}| \le \nu |w^s| \ , & |\widehat{w}^{uu}| \ge \frac{1}{\mu_0} |w^u| \end{array} . \tag{3.6}$$

First, by Lemma 2.1 the decomposition of $\widetilde{w}$ at $\phi(x_k)$:

$$\widetilde{w} = \widetilde{w}^{ss} + \widetilde{w}^{su} + \widetilde{w}^{us} + \widetilde{w}^{uu} \ ,$$

where $\widetilde{w}^{\sigma\tau} = P^\sigma_{\phi(x_k)} D\phi(x_k) w^\tau$, $\sigma, \tau \in \{s, u\}$ satisfies

$$\begin{array}{ll} |\widetilde{w}^{ss}| \le \mu |w^s| \ , & |\widetilde{w}^{su}| \le \frac{\nu}{2} |w^u| \\ |\widetilde{w}^{us}| \le \frac{\nu}{2} |w^s| \ , & |\widetilde{w}^{uu}| \ge \frac{1}{\mu} |w^u| \end{array} .$$

Then (3.6) follows from (3.5), since $|x_{k+1} - \phi(x_k)| \le \delta < \delta_2$.

Now since $(DF(0)v)_{k+1} = D\phi(x_k)v_k$ we get, according to the decomposition of l_p, that

$$DF(0) = \begin{bmatrix} A^{ss} & A^{su} \\ A^{us} & A^{uu} \end{bmatrix} \ ,$$

where

$$\|A^{ss}\| \le \mu_0 \ , \quad \|(A^{uu})^{-1}\| \le \mu_0 \ , \quad \|A^{su}\|, \|A^{us}\| \le \nu \ .$$

Hence $DF(0) - I$ is invertible and $\big\|(I - DF(0))^{-1}\big\| \le R$.

Now we solve the equation $F(v) = v$, which is equivalent to (3.3). Set

$$G(v) = (I - DF(0))^{-1}(F(v) - DF(0)v) \, .$$

Then $F(v) = v$ is clearly equivalent to $v = G(v)$.

Define $H(v) = F(v) - DF(0)v$, i.e., $H(v)_k = [\phi(x_{k-1}) - x_k] + \psi_{k-1}(v_{k-1})$. Then

$$\|H(0)\|_p = \Big(\sum_{k\in\mathbf{Z}} |d(\xi)_k|^p\Big)^{1/p} \le \delta \ . \tag{3.7}$$

It follows from the properties of ψ that, for $v_1,\, v_2 \in B_p(\delta_1)$,

$$\|H(v^1) - H(v^2)\|_p \le \beta(\max(\|v^1\|_\infty, \|v^2\|_\infty))\, \|v^1 - v^2\|_p \ . \tag{3.8}$$

It follows from (3.7) and (3.8) that $\|G(0)\|_p \le R\delta$ and

$$\|G(v^1) - G(v^2)\|_p \le R\beta(\max_{i=1,2} \|v^i\|_\infty)\|v^1 - v^2\|_p \ .$$

Take $\delta_3 \in (0, \delta_1)$ such that $R\beta(\delta_3) < \frac{1}{2}$. Since $\|v\|_\infty \le \|v\|_p$ we have

$$\|G(v^1) - G(v^2)\|_p \le \tfrac{1}{2}\|v^1 - v^2\|_p$$

for $v^1,\, v^2 \in B_p(\delta_3)$. That is G contracts on the ball $B_p(\delta_3)$. Set

$$\delta_0 = \min(\delta_2, \tfrac{\delta_3}{2R}) \ , \quad L = 2R \ .$$

Then if $\delta < \delta_0$, we have for $v \in B_p(L\delta)$:

$$\|G(v)\|_p \le \|G(0)\|_p + \|G(v) - G(0)\|_p \le R\delta + \tfrac{1}{2}\, 2R\delta = L\delta \ .$$

Hence G is a contraction of $B_p(L\delta)$ and G has a unique fixed point $v^* = \{v_k^*\}_{k\in\mathbf{Z}}$ with $\|v^*\|_p \le L\delta$. It follows from our construction that $\{y_k\}_{k\in\mathbf{Z}} = \{x_k + v_k^*\}_{k\in\mathbf{Z}}$ is a trajectory, i.e., $y_k = \phi^k(y_0)$ and

$$\Big(\sum_{k\in\mathbf{Z}} |x_k - \phi^k(y_0)|^p\Big)^{1/p} = \Big(\sum_{k\in\mathbf{Z}} |v_k^*|^p\Big)^{1/p} \le L\delta \ .$$

This completes the proof. $\square$

Remark. If $\xi = \{x_k\}_{k\ge 0}$ is a sequence such that $\|d^+(\xi)\|_p \le \delta$ with $\delta < \delta_0/2$, then assuming also that $d(x_0, \Lambda) < \delta$ we can take $\widetilde{x}_0 \in \Lambda$ such that $|\widetilde{x}_0 - x_0| < \delta$ and extend ξ to $\widetilde{\xi} = \{x_k\}_{k\in\mathbf{Z}}$ by defining $x_k = \phi^k(\widetilde{x}_0)$ for $k < 0$. Then $\widetilde{\xi}$ satisfies $\|d(\widetilde{\xi})\|_p \le 2\delta$, and applying Theorem 3.1 we can find $\eta = \{y_k\}_{k\in\mathbf{Z}}$ – a trajectory of ϕ – such that $\|\eta - \widetilde{\xi}\|_p \le 2L\delta$, which implies $\|\Delta^+(y_0, \xi)\|_p \le 2L\delta$.

Remark. In [4] Easton studied the following property of a dynamical system ϕ on a metric space (X, d) : given $\varepsilon > 0$ there exists a $\delta > 0$ such that if a sequence of points $\{x_k\}_{k\in\mathbf{Z}}$ satisfies

$$\textstyle\sum_k d(x_{k+1}, \phi(x_k)) < \delta$$

then there exists a point x such that

$$\textstyle\sum_k d(x_k, \phi^k(x)) < \varepsilon \ .$$

This was called the strong shadowing property. In case $p=1$ Theorem 3.1 shows that a diffeomorphism has the strong shadowing property in a neighbourhood of a hyperbolic set.

If we take $p=\infty$ the shadowing can be asked in two ways: either considering bounded sequences or sequences with zero limits, both with the standard supnorm topology. The first case is the basic shadowing result (Lemma 2.2). For the latter we get

Proposition 3.2 *There exists a neighbourhood U of Λ and L, $\delta_0>0$ such that if for a sequence $\xi=\{x_k\}_{k\in\mathbf{Z}}\subset U$ we have $\|d(\xi)\|_\infty<\delta_0$ and $\lim_{|k|\to\infty} d(\xi)_k=0$, then there exists a unique x^* such that $\|\Delta(x^*,\xi)\|_\infty\le L\|d(\xi)\|_\infty$ and $\lim_{|k|\to\infty}|x_k-\phi^k(x^*)|=0$.*

Proof. In the proof of previous theorem take, in the place of l_p, the Banach space c_0 of sequences $\{u_k\}_{k\in\mathbf{Z}}\subset \mathbb{R}^n$ such that $\lim_{|k|\to\infty}|u_k|=0$ with $\|u\|=\max_k|u_k|$. Otherwise the proof is similar. □

Next we will consider weighted analogues of inequalities (1.1) and (1.2)

$$\Big(\sum_{k\in\mathbf{Z}}\rho^{|k|}|x_{k+1}-\phi(x_k)|^p\Big)^{1/p}\le\delta\ , \tag{3.9}$$

$$\Big(\sum_{k\in\mathbf{Z}}\rho^{|k|}|\phi^k(x)-x_k|^p\Big)^{1/p}\le L\delta \tag{3.10}$$

with some $\rho>1$.

Theorem 3.3 *Assume that*

$$\rho^{1/p}\lambda<1\,. \tag{3.11}$$

Then there exists a neighbourhood U of Λ and numbers L , $\delta_0>0$ such that if a sequence satisfies (3.9) with $\delta<\delta_0$, then there exists a unique x^* such that (3.10) holds.

Proof. Take a neighbourhood U_0 of Λ and an adapted metric such that (3.11) still holds with the corresponding slightly increased λ. Take $\mu>\lambda$ and $\gamma\in(1,2)$ such that $\mu_0=\gamma\rho^{1/p}\mu<1$. For this μ_0 take the corresponding ν, R. (See the beginning of the proof of Theorem 3.1.)

Now the proof goes parallel to that of Theorem 3.1. The only difference is in obtaining a bound for $\|I-DF(0)\|$, where F is defined as before, but now in the Banach space of sequences with norm

$$\|v\|_{\rho,p}=\Big(\sum_{k\in\mathbf{Z}}\rho^{|k|}|v|^p\Big)^{1/p}\ .$$

Instead of inequalities (3.6) we now take small enough $\delta_2 > 0$ so that for $\widetilde{w} = D\phi(x_k)w$ we get

$$(3.12) \qquad \begin{aligned} |\widehat{w}^{ss}| \le \gamma\mu |w^s| \ , \quad |\widehat{w}^{su}| \le \tfrac{\nu}{\rho^{1/p}} |w^u| \\ |\widehat{w}^{us}| \le \tfrac{\nu}{\rho^{1/p}} |w^s| \ , \quad |\widehat{w}^{uu}| \ge \tfrac{1}{\gamma\mu} |w^u| \end{aligned}$$

Then for a sequence $\widetilde{v} = \left\{ D\phi(x_{k-1}) v^{\,s}_{k-1} \right\}_{k\in\mathbf{Z}}$, where $\widetilde{v}_k = \widetilde{v}^{\,s}_{k} + \widetilde{v}^{\,u}_{k}$ with

$$v^{\,s}_{k}\,,\ \widetilde{v}^{\,s}_{k} \in E^s_{x_k} \ , \quad \widetilde{v}^{\,u}_{k} \in E^u_{x_k}$$

we get

$$\sum_{k\in\mathbf{Z}} \rho^{|k|} |\widetilde{v}^{\,s}_{k}|^p \le \sum_{k\in\mathbf{Z}} \rho^{|k|} (\gamma\mu |v^{\,s}_{k-1}|)^p = (\gamma\mu)^p \sum_{k\in\mathbf{Z}} \rho^{|k+1|} |v^{\,s}_{k}|^p \le (\gamma\mu)^p \rho \sum_{k\in\mathbf{Z}} \rho^{|k|} |v^{\,s}_{k}|^p,$$

i.e., $\|\widetilde{v}^{\,s}\|_{\rho,p} \le \rho^{1/p}\gamma\mu \, \|v^{\,s}\|_{\rho,p}$. Hence for

$$DF(0) = \begin{bmatrix} A^{ss} & A^{su} \\ A^{us} & A^{uu} \end{bmatrix}$$

we get $\|A^{ss}\| \le \rho^{1/p}\gamma\mu = \mu_0$. Inequalities

$$\|(A^{uu})^{-1}\| \le \mu_0 \ , \quad \|A^{su}\| \, , \, \|A^{us}\| \le \nu$$

are obtained similarly. The rest of the proof is the same as in Theorem 3.1. □

Clearly a result for one–sided sequences similar to that of the remark after Theorem 3.1 is valid also here.

Now we show that under certain conditions on the weights we get limit weighted shadowing without assuming the invariant set to be hyperbolic. First we study one-sided sequences and "strong enough" weights.. For an increasing sequence $r = \{r_k\}_{k\ge 0}$ of positive weights and $p \ge 1$ denote $l_{r,p}$ the Banach space of sequences with norm

$$\|v\|_{r,p} = \Big(\sum_{k\ge 0} r_k |v_k|^p \Big)^{1/p} \ .$$

Theorem 3.4 *Let ϕ be a C^1–diffeomorphism of $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ bounded. Assume the weight sequence satisfies $r_{k+1} \ge \rho r_k$ for $k \ge k_0$, where*

$$(3.13) \qquad \rho^{1/p} > M = \max\big(\, 1 \, , \, \sup_{x\in U} \|D\phi(x)^{-1}\| \, \big) \ .$$

Then there exist $L,\, \delta_0 > 0$ such that if $\xi = \{x_k\}_{k\ge 0} \subset U$ satisfies $\|d^+(\xi)\|_{r,p} < \delta_0$, then there exists a unique x^* such that

$$(3.14) \qquad \left\| \Delta^+(x^*, \xi) \right\|_{r,p} \le L \left\| d^+(\xi) \right\|_{r,p} \ .$$

Proof. First note that taking $c = \min\limits_{0\le j\le k_0} \rho^{-j} r_j > 0$ we have $r_k \ge c\rho^k$. As in the proof of previous theorems we look for a sequence $\{v_k\}_{k\ge 0} \in l_{r,p}$ satifying

$$\phi(x_k + v_k) = x_{k+1} + v_{k+1} \ , \ \ k \ge 0 \ ,$$

i.e., $F(v) = Sv$, where $F(v)_k = \phi(x_k + v_k) - x_{k+1}$ and S is the left shift:

$$S(v_0, v_1, \ldots) = (v_1, v_2, \ldots) \ .$$

Let us first show that $S - DF(0)$ has a bounded right inverse in $l_{r,p}$. For this we will need the following: if $\{a_j\}_{j\geq 0}$ is a nonnegative sequence, and $p, \eta > 1$, then

$$\sum_{j\geq 0} a_j \leq C(\eta,p)\Big(\sum_{j\geq 0} \eta^j a_j^p\Big)^{1/p} , \quad \text{where} \quad C(\eta,p) = \left(\frac{\eta^{\frac{1}{p-1}}}{\eta^{\frac{1}{p-1}} - 1}\right)^{1-\frac{1}{p}} . \tag{3.15}$$

Indeed, by Hölder's inequality:

$$\sum_{j\geq 0} a_j = \sum_{j\geq 0} \eta^{j/p} a_j \ \eta^{-j/p} \leq \Big(\sum_{j\geq 0} \eta^j a_j^p\Big)^{1/p} \Big(\sum_{j\geq 0} (\eta^{-j/p})^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}$$

from which (3.15) follows. For $p = 1$ the inequality (3.15) holds trivially with $C(\eta, 1) = 1$

Equation $Sv - DF(0)v = w$ is equivalent to

$$v_{k+1} = D\phi(x_k)v_k + w_k \ , \quad k \geq 0 \ . \tag{3.16}$$

Set $D_{k,k} = I$ and $D_{j+1,k} = D\phi(x_j)D\phi(x_{j-1})\ldots D\phi(x_k)$. We claim that the sequence $\{v_k\}_{k\geq 0}$ defined by

$$v_k = -\sum_{j=k}^{\infty} D^{-1}_{j+1,k} w_j \ , \quad k \geq 0 \tag{3.17}$$

satisfies (3.16). To show the convergence of (3.17) take $\widetilde{\rho} = (M^p + \rho)/2$ and $\eta = \widetilde{\rho}/M^p$. Then by (3.13) and (3.15)

$$\begin{aligned} |v_k| &\leq \sum_{j=k}^{\infty} |D^{-1}_{j+1,k} w_j| \ \leq \ \sum_{j=k}^{\infty} M^{j+1-k} |w_j| \ = \ M \sum_{j=0}^{\infty} M^j |w_{k+j}| \\ &\leq MC(\eta,p)\Big(\sum_{j=0}^{\infty} \Big(\tfrac{\widetilde{\rho}}{M^p}\Big)^j \Big(M^j |w_{k+j}|\Big)^p\Big)^{1/p} \\ &= MC(\eta,p)\Big(\sum_{j=k}^{\infty} \widetilde{\rho}^{\,j-k} |w_j|^p\Big)^{1/p} . \end{aligned} \tag{3.18}$$

Since $\widetilde{\rho} < \rho$ and $\rho^{j-k} \leq \frac{r_j}{c}$, the last line of (3.18) is at most $\frac{M}{c} C(\eta,p) \, \|w\|_{r,p}$. Hence the sums of (3.17) converge. Further,

$$\begin{aligned} v_{k+1} &= -\sum_{j=k+1}^{\infty} D^{-1}_{j+1,k+1} w_j = -D_{k+1,k} \sum_{j=k+1}^{\infty} (D_{j+1,k+1} D_{k+1,k})^{-1} w_j \\ &= -D\phi(x_k) \sum_{j=k}^{\infty} D^{-1}_{j+1,k} w_j + w_k = D\phi(x_k) v_k + w_k \ , \end{aligned}$$

i.e., v satisfies (3.16). A bound for the norm of v comes using (3.18):

$$\begin{aligned}\sum_{k=0}^{\infty} r_k\,|v_k|^p &\le M^p C(\eta,p)^p \sum_{k=0}^{\infty} r_k \sum_{j=k}^{\infty} \widetilde{\rho}^{\,j-k}|w_j|^p\\ &= M^p C(\eta,p)^p \sum_{j=0}^{\infty}\Big(\sum_{k=0}^{j}\frac{r_k}{r_j}\widetilde{\rho}^{\,j-k}\Big) r_j|w_j|^p\ .\end{aligned}$$

Here we have:

$$\begin{aligned}\sum_{k=0}^{j}\frac{r_k}{r_j}\widetilde{\rho}^{\,j-k} &\le \sum_{k=0}^{k_0-1}\frac{r_{k_0}}{r_j}\widetilde{\rho}^{\,j-k}+\sum_{k=k_0}^{j}\frac{r_k}{r_j}\widetilde{\rho}^{\,j-k}\\ &\le \sum_{k=0}^{k_0-1}\frac{\widetilde{\rho}^{\,j-k}}{\rho^{j-k_0}}+\sum_{k=k_0}^{j}\frac{\widetilde{\rho}^{\,j-k}}{\rho^{j-k}} \le \frac{\widetilde{\rho}\rho^{k_0}}{\widetilde{\rho}-1}+\frac{\rho}{\rho-\widetilde{\rho}} = C_0\ .\end{aligned}$$

Hence $Sv - DF(0)v = w$ has a solution v depending linearly on w such that $\|v\|_{r,p} \le MC(\eta,p)C_0^{1/p}\ \|w\|_{r,p}$, i.e., $S-DF(0)$ has a bounded right inverse. It follows that $S-DF(0)$ is invertible, provided it is one–to–one. So, assume v is such that $Sv-DF(0)v=0$ and $v_m \ne 0$ for some m. Then for $k>m$ we have

$$v_k = D\phi(x_{k-1})v_{k-1}\ , \qquad |v_k| \ge M^{-1}\,|v_{k-1}| \ge M^{m-k}\,|v_m|\ ,$$

and

$$\sum_{k=m}^{\infty} r_k\,|v_k|^p \ge c\,|v_m|^p\ M^{mp}\sum_{k=m}^{\infty}(\rho/M^p)^k = \infty$$

so that v is not in $l_{r,p}$. Hence $S-DF(0)$ has a bounded inverse in $l_{r,p}$. Now $F(v)=Sv$ is equivalent to

$$v = (S-DF(0))^{-1}(F(v)-DF(0)v)\ .$$

Solving this goes as in the proof of Theorem 3.1. □

Refined result for positive pseudo–orbits

We saw that weighted l_p–shadowing of pseudo–orbits can be obtained for slowly increasing weights in the hyperbolic situation (Thm 3.3) or generally for strong enought weights for positive pseudo–orbits. Then the question arises: *which (geometric) speeds of decreasing one–step errors can be seen in the shadowing errors?*

Here we need the concept of Sacker–Sell spectrum. Let Λ be a compact invariant set of a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1. Sacker and Sell [10] studied the so-called spectrum $\Sigma(\phi,\Lambda)$ and the resolvent set $\mathcal{R}(\phi,\Lambda) = \mathbb{R}\setminus\Sigma(\phi,\Lambda)$ of the pair ϕ,Λ. Let us formulate one of their main results.

We say that the set Λ is invariantly connected if it cannot be represented as the union of two disjoint nonempty compact invariant sets.

Proposition 3.5 *([10]). Assume that Λ is invariantly connected. Then* *a) the spectrum* $\Sigma(\phi,\Lambda)$ *is the union of* $k \le n$ *compact intervals:*

$$\Sigma(\phi,\Lambda) = [a_1,b_1] \cup \ldots \cup [a_k,b_k]$$

b) for any $\rho \in \mathcal{R}(\phi,\Lambda)$ *there exist constants* $C > 0$, $\lambda \in (0,1)$ *and a continuous family of linear subspaces* E_x^-, $E_x^+ \subset \mathbb{R}^n$, $x \in \Lambda$, *such that*

(b.1) $E_x^- \oplus E_x^+ = \mathbb{R}^n$

(b.2) $D\phi(x)E_x^\pm = E_{\phi(x)}^\pm$

(b.3)
$$\begin{aligned} \rho^k \left|D\phi^k(x)v\right| &\le C\lambda^k |v| \ , \ \ v \in E_x^-, \ k \ge 0 \\ \rho^{-k} \left|D\phi^{-k}(x)v\right| &\le C\lambda^k |v| \ , \ \ v \in E_x^+, \ k \ge 0 \end{aligned}$$

Remark 3.1 *In [10] the spectrum is defined in such a way that analogues of (b.3) hold for* $\exp(-k\mu)\left|D\phi^k(x)v\right|$, *hence* $\Sigma(\phi,\Lambda)$ *and the original Sacker – Sell spectrum are related by the transformation* $\rho \mapsto \exp(-\mu)$. *Of course, this does not change the geometry of the spectra.*

Remark 3.2 *Relations between* $\Sigma(\phi,\Lambda)$ *and the Mather spectrum [7] were studied in [6].*

For $\rho \ge 1$, $p \ge 1$ we work again with the Banach space $l_{\rho,p}$ of positive sequences with the norm
$$\|v\|_{\rho,p} = \Big(\sum_{k\ge 0} \rho^k \, |v_k|^p\Big)^{1/p}.$$

Theorem 3.6 *Assume that* Λ *is invariantly connected and that for some* ρ, $p \ge 1$ *we have* $\widetilde{\rho} := \rho^{1/p} \in \mathcal{R}(\phi,\Lambda)$. *Then there exists a neighbourhood* W *of* Λ *and numbers* δ_0, $L > 0$ *such that if a sequence of points* $\xi = \{x_k\}_{k\ge 0} \subset W$ *satisfies*
$$\left\|d^+(\xi)\right\|_{\rho,p} \le \delta_0 \ ,$$
then there is a point x^* *such that*

$$\left\|\Delta^+(x^*,\xi)\right\|_{\rho,p} \le L \left\|d^+(\xi)\right\|_{\rho,p}. \tag{3.19}$$

In the proof we use the following two lemmas:

Lemma 3.1 *Assume that* $\widetilde{\rho} \in \mathcal{R}(\phi,\Lambda)$, $\lambda \in (0,1)$, *and* C *are as in Prop. 3.5. Then, for given* $\varepsilon > 0$, $\mu_0 \in (\lambda,1)$ *there exists, in a neighbourhood* W *of* Λ, *a Lyapunov norm* $|\cdot|_x$ *and a (not necessarily* $D\phi$ *–invariant) extension of families* E_x^-, E_x^+ *to* W, *and positive numbers* M, δ_1 *such that for* x, $x' \in W$, $d(x',\phi(x)) < \delta_1$, *we have*

$$\widetilde{\rho}\,|P_{x'}^- D\phi(x)v|_{x'} \le (1+\varepsilon)\mu_0 |v|_x \ , \qquad \widetilde{\rho}\,|P_{x'}^+ D\phi(x)v|_{x'} \le \varepsilon |v|_x \ , \qquad v \in E_x^- \tag{3.20}$$

$$\widetilde{\rho}\,|P_{x'}^+ D\phi(x)v|_{x'} \ge \frac{1}{\mu_0(1+\varepsilon)} |v|_x \ , \qquad \widetilde{\rho}\,|P_{x'}^- D\phi(x)v|_{x'} \le \varepsilon |v|_x \ , \qquad v \in E_x^+ \ , \tag{3.21}$$

where P_x^- *is the projection onto* E_x^- *along* E_x^+ *and* $P_x^+ = I - P_x^-$, $x \in W$. *Further,*

$$\tfrac{1}{M}|v|_x \le |v| \le M|v|_x \ , \quad x \in W \, , \ v \in \mathbb{R}^n . \tag{3.22}$$

Proof. We construct the Lyapunov norm in the usual way as follows. For $x \in \Lambda$, $v = v^- + v^+ \in E_x^- \oplus E_x^+$ set

$$|v|_x = \Big(|v^-|_x^2 + |v^+|_x^2\Big)^{1/2} ,$$

where

$$|v^-|_x = \sum_{j=0}^{N} \widetilde{\rho}^j \mu_0^{-j} \, |D\phi^j(x)v^-| , \quad |v^+|_x = \sum_{j=0}^{N} \widetilde{\rho}^{-j} \mu_0^{-j} \, |D\phi^{-j}(x)v^+| ,$$

and N is such that $C(\lambda/\mu_0)^{N+1} < 1$. Then for $v^- \in E_x^-$ we get:

$$\begin{aligned}
\widetilde{\rho}\,|D\phi(x)v^-|_{\phi(x)} &= \sum_{j=0}^{N} \widetilde{\rho}^{j+1} \mu_0^{-j} \, |D\phi^j(\phi(x)) D\phi(x) v^-| \\
&= \mu_0 \left(\sum_{j=1}^{N} \widetilde{\rho}^j \mu_0^{-j} \, |D\phi^j(x)v^-| \; + \; \widetilde{\rho}^{N+1} \mu_0^{-N-1} \, |D\phi^{N+1}(x)v^-| \right) \\
&\le \mu_0 \left(\sum_{j=1}^{N} \widetilde{\rho}^j \mu_0^{-j} \, |D\phi^j(x)v^-| \; + \; \mu_0^{-N-1}\, C\, \lambda^{N+1} |v^-| \right) \le \mu_0 |v^-|_x \; .
\end{aligned}$$

Similarly, for $v^+ \in E_x^+$ we get:

$$\begin{aligned}
\widetilde{\rho}\,|D\phi(x) \quad v^+|_{\phi(x)} &= \sum_{j=0}^{N} \widetilde{\rho}^{-j+1} \mu_0^{-j} \, |D\phi^{-j}(\phi(x)) D\phi(x) v^+| \\
&= \frac{1}{\mu_0} \left(\sum_{j=0}^{N-1} \widetilde{\rho}^{-j} \mu_0^{-j} \, |D\phi^{-j}(x)v^-| \; + \; \widetilde{\rho}\mu_0 \, |D\phi(x)v^+| \right) \\
&\ge \frac{1}{\mu_0} \left(\sum_{j=0}^{N-1} \widetilde{\rho}^{-j} \mu_0^{-j} \, |D\phi^{-j}(x)v^-| \; + \; \frac{\widetilde{\rho}\mu_0}{C\lambda^{N+1}\widetilde{\rho}^{N+1}} \, |D\phi^{-N}(x)v^+| \right) \ge \frac{1}{\mu_0} |v^+|_x \; .
\end{aligned}$$

These estimates allow us to to extend $|\cdot|_x$ to a neighbourhood of Λ so that (3.20) and (3.21) hold. For details see: [5] and the proof of Thm. 3.1. $\square$

Lemma 3.2 *Let A be a linear map between Banach spaces V_0 and V_1, which has a right inverse $A^\dagger : V_1 \to V_0$ (i.e., $AA^\dagger = I$ on V_1). If $B : V_0 \to V_1$ is linear such that $\|A - B\|\,\|A^\dagger\| < 1$, then B is right invertible with*

$$\|B^\dagger\| \le \frac{\|A^\dagger\|}{1 - \|A - B\|\,\|A^\dagger\|} \; .$$

Proof. Since $\|(A - B)A^\dagger\| < 1$, the series

$$\widetilde{B} = A^\dagger + A^\dagger(A - B)A^\dagger + A^\dagger\Big((A - B)A^\dagger\Big)^2 + \ldots$$

converges and

$$\begin{aligned}
&B\widetilde{B} = (A + (B - A))\Big(A^\dagger + A^\dagger(A - B)A^\dagger + A^\dagger(A - B)A^\dagger(A - B)A^\dagger + \ldots\Big) \\
&= I + (A - B)A^\dagger + (A - B)A^\dagger(A - B)A^\dagger + \ldots \\
&+(B - A)A^\dagger + (B - A)A^\dagger(A - B)A^\dagger + (B - A)A^\dagger(A - B)A^\dagger(A - B)A^\dagger + \ldots = I \; ,
\end{aligned}$$

i.e., $B^\dagger = \widetilde{B}$ is a right inverse of B. $\square$

Proof. (of Thm. 3.6). Take $\varepsilon > 0$ such that $\mu := (1+\varepsilon)\mu_0 < 1$ and $\dfrac{\varepsilon}{1-(1+\varepsilon)\mu_0} < 1$.

As before, given a pseudotrajectory $\xi \subset W$ with $\|d(\xi)\|_{\rho,p} \le \delta_1$, we can write the equation for a shadowing trajectory $\xi + v$ as

$$v_{k+1} = \phi(x_k + v_k) - x_{k+1}\,, \qquad k = 0, 1, 2, \ldots,$$

i.e., $Sv = F(v)$, where $S, F \,:\, l_{\rho,p} \to l_{\rho,p}$

$$(Sv)_k = v_{k+1}\,, \qquad F(v)_k = \phi(x_k + v_k) - x_{k+1}\,.$$

First we show that $S - DF(0)$ has a right inverse, which is uniformly bounded for $\xi \subset W$ satisfying $\|d(\xi)\|_{\rho,p} \le \delta_1$.

Consider S and F as maps $\mathcal{E} \to \widetilde{\mathcal{E}}$, where $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are the Banach spaces of sequences $\{v_k\}_{k\ge 0}$ with norms

$$\|v\|^p = \sum_{k=0}^{\infty} \rho^k \Big(|P^-_{x_k} v_k|^p_{x_k} + |P^+_{x_k} v_k|^p_{x_k} \Big)\,, \qquad \|v\|^p_\sim = \sum_{k=0}^{\infty} \rho^{k+1} \Big(|P^-_{x_{k+1}} v_k|^p_{x_{k+1}} + |P^+_{x_{k+1}} v_k|^p_{x_{k+1}} \Big)\,,$$

respectively. By (3.22) these norms are equivalent to that of $l_{\rho,p}$, i.e., there exists $C > 0$ independent of ξ such that

$$\frac{1}{C}\, \|v\|_\sim \le \|v\|_{\rho,p} \le C\, \|v\|\,. \tag{3.23}$$

Denote $\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$, $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}^- \oplus \widetilde{\mathcal{E}}^+$, where

$$\mathcal{E}^\pm = \Big\{ v \in \mathcal{E} \,|\, v_k \in E^\pm_{x_k}\,,\ k \ge 0 \Big\}\,, \qquad \widetilde{\mathcal{E}}^\pm = \Big\{ v \in \widetilde{\mathcal{E}} \,|\, v_k \in E^\pm_{x_{k+1}}\,,\ k \ge 0 \Big\}\,,$$

Split S and $DF(0)$ accordingly:

$$S = \begin{bmatrix} S_{--} & 0 \\ 0 & S_{++} \end{bmatrix}\,, \qquad DF(0) = \begin{bmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{bmatrix}\,.$$

S_{++} has norm 1 and S_{--} has a right inverse $S^\dagger_{--} \,:\, \widetilde{\mathcal{E}}^- \to \mathcal{E}^-$

$$(S^\dagger_{--} u)_k = \begin{cases} 0 & \text{for } k = 0 \\ u_{k-1} & \text{for } k \ge 1 \end{cases}\,,$$

which also has norm 1.

By (3.20) we get for $v \in \mathcal{E}^-$

$$\|A_{--} v\|^p_\sim = \sum_{k=0}^{\infty} \rho^{k+1} |P^-_{x_{k+1}} D\phi(x_k) v_k|^p_{x_{k+1}} \le (1+\varepsilon)^p \mu_0^p \sum_{k=0}^{\infty} \rho^k |v_k|^p_{x_k} = (1+\varepsilon)^p \mu_0^p\, \|v\|^p\,,$$

i.e., $\|A_{--}\| \le \mu$, and

$$\|A_{+-} v\|^p_\sim = \sum_{k=0}^{\infty} \rho^{k+1} |P^+_{x_{k+1}} D\phi(x_k) v_k|^p_{x_{k+1}} \le \varepsilon^p \sum_{k=0}^{\infty} \rho^k |v_k|^p_{x_k} = \varepsilon^p\, \|v\|^p\,,$$

implying $\|A_{+-}\| \leq \varepsilon$. Since $\|A_{--}\| < 1$, and $\|S_{--}^\dagger\| = 1$, we get by Lemma 3.2 that $S_{--} - A_{--}$ has a right inverse $\widetilde{\mathcal{E}}^- \to \mathcal{E}^-$ with bound

$$\|(S_{--} - A_{--})^\dagger\| \leq \frac{\|S_{--}^\dagger\|}{1 - \|A_{--}\|\,\|S_{--}^\dagger\|} \leq \frac{1}{1-\mu} \ . \tag{3.24}$$

Similarly, for $v \in \mathcal{E}^+$ we get

$$\|A_{++}\| \geq 1/\mu \quad \text{and} \quad \|A_{-+}\| \leq \varepsilon \ .$$

It is easy to see that A_{++} maps $\mathcal{E}^+$ onto $\widetilde{\mathcal{E}}^+$ so that A_{++} is invertible with $\|A_{++}^{-1}\| \leq \mu$. Again by Lemma 3.2 $S_{++} - A_{++}$ has a right inverse $\widetilde{\mathcal{E}}^+ \to \mathcal{E}^+$ with bound

$$\|(S_{++} - A_{++})^\dagger\| \leq \frac{\mu}{1-\mu} \ . \tag{3.25}$$

By (3.24) and (3.25) $S - \begin{bmatrix} A_{--} & 0 \\ 0 & A_{++} \end{bmatrix}$ has right inverse which has norm bound $1/(1-\mu)$. Further, $\begin{bmatrix} 0 & A_{-+} \\ A_{+-} & 0 \end{bmatrix}$ has norm at most ε. So, by Lemma 3.2, $S - DF(0)$ has a right inverse $\widetilde{\mathcal{E}} \to \mathcal{E}$ with norm

$$\|(S - DF(0))^\dagger\| \leq \frac{1/(1-\mu)}{1 - \varepsilon/(1-\mu)} = \frac{1}{1-\mu-\varepsilon} \ . \tag{3.26}$$

Hence by (3.23) it is right invertible from $l_{\rho,p}$ to $l_{\rho,p}$ with norm

$$\|(S - DF(0))^\dagger\|_{\rho,p} \leq R = \frac{C^2}{1-\mu-\varepsilon} \ .$$

This bound does not depend on ξ, provided $\|d(\xi)\|_{\rho,p} \leq \delta_1$.

Consider equation $v = G(v)$, where

$$G(v) = (S - DF(0))^\dagger (F(v) - DF(0)v) \ .$$

By multiplication with $S - DF(0)$ we see that a solution of this also satisfies $Sv = F(v)$. As in the proof of Thm 3.1 we can take $\delta_0 \in (0, \delta_1)$ such that G has Lipschitz constant $< \frac{1}{2}$ in the ball $\|v\|_{\rho,p} < 2R\delta_0$ uniformly in ξ satisfying $\|d(\xi)\|_{\rho,p} \leq \delta_1$. If now $\delta = \|d(\xi)\|_{\rho,p} \leq \delta_0$, then G contracts the ball $\|v\|_{\rho,p} < 2R\delta$, i.e., equation $v = G(v)$, has a solution there. □

Remark. Note that, in contrast to Thm. 3.4, the point x^* is not necessarily unique. It is not difficult to show that x^* is unique if $\Sigma(\phi, \Lambda) \cap [\widetilde{\rho}, \infty) = \emptyset$ (in this case $E_x^+ = R^n$ for $x \in \Lambda$).

4 Example: limit shadowing of periodic flows

Consider a system of differential equations

$$\dot{x}(t) = f(t, x(t)) \ , \qquad x(t) \in \mathbb{R}^n \ . \tag{4.1}$$

Assume that f and $\partial_x f$ are continuous and bounded and that f is 1–periodic in t. Denote by $x(t, t_0, x_0)$ the solution of (4.1) with initial value $x(t_0) = x_0$ and let ϕ be the Poincaré map of (4.1), that is,

$$\phi(x_0) = x(1, 0, x_0) \ .$$

Our assumptions on f imply that ϕ is a diffeomorphism of class C^1.

Assume that ϕ has a hyperbolic attractor Λ. We consider two types of approximations of ϕ: using Picard-Lindelöf –iterations and one–step discretizations.

Approximation with Picard-Lindelöf –iterations

Denote by T the Picard operator on $C[0,1]$:

$$Tx(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau$$

and set $\psi_j(x_0) = (T^j \widetilde{x}_0)(1)$, where $\widetilde{x}_0(t) = x_0$, $t \in [0,1]$. The following is a standard result:

$$|\psi_j(x_0) - \phi(x_0)| \le C \frac{M^{j+1}}{(j+1)!} \ , \tag{4.2}$$

where $M = \sup_{t,x} |f_x(t,x)|$ and $C = \frac{e^M}{M} \sup_{t,x} |f(t,x)|$.

Let $x_{k+1} = \psi_{j_k}(x_k)$, $k \ge 0$ with x_0 close to Λ, j_0 big enough, $\{j_k\}_{k \ge 0}$ increasing, and $\lim_{k\to\infty} j_k = \infty$. Then by Theorem 2.1 there exists x^* such that $\lim_{k\to\infty} \left|\phi^k(x^*) - x_k\right| = 0$. In order to get l_p–shadowing from Theorem 3.1 , one needs

$$\sum_{k=0}^{\infty} \left(\frac{M^{j_k}}{(j_k+1)!} \right)^p < \infty \ ,$$

for which one easily obtains from the Stirling formula that growth $j_k \ge j_0 + \frac{1}{p}\ln(k)$ is enough. For weights $r_k = \rho^k$ (see Theorem 3.3) we can set $j_k \ge j_0 + \frac{k \ln(\rho)}{p}$.

On the other hand, if j_k's increase for example like $j_k = j_0 + k$, then using Theorem 3.4 with weights

$$r_k = \left(\frac{(j_k+1)!}{k^{1+\varepsilon} M^{j_k+1}} \right)^p$$

we obtain that there exists a trajectory, which is approached with essentially the speed of decrease of the approximation errors.

Approximation with one–step discretizations

Assuming f is smooth enough, any standard one–step discretization (e.g. a Runge–Kutta method) gives a map

$$\Psi_h(t_0, x_0) = (\, t_0 + h \, , \, x(t_0 + h, t_0, x_0) + O(h^{q+1}) \,) \ ,$$

where $q \ge 1$ is the order of the method.

Take an increasing sequence of natural numbers $\{n_k\}_{k\geq 0}$, x_0 close to Λ, set $h_k = 1/n_k$, and define a sequence $\{x_k\}_{k\geq 0}$ by

$$\Psi_{h_k}^{n_k}(0, x_k) = (1, x_{k+1}) \ .$$

Then $|\phi(x_k) - x_{k+1}| \leq C n_k^{-q}$.
Again, if $\lim_{k\to\infty} n_k = \infty$ and n_0 is big enough, then by Theorem 2.1 there exists x^* such that $\lim_{k\to\infty} \left|\phi^k(x^*) - x_k\right| = 0$. For l_p–shadowing from Theorem 3.1 , it suffices to have $n_k \geq n_0 + k^{\alpha/pq}$ with some $\alpha > 1$. For weights $r_k = \rho^k$ (see Theorem 3.3) we can set $n_k \geq n_0 + \rho^{k/pq}\, k^{\alpha/pq}$.

References

[1] E. Akin. *The General Topology of Dynamical Systems.* AMS, Grad. Stud. in Math., vol.1, 1993.

[2] D. V. Anosov. *Об одном классе инвариантных множеств гладких динамических систем* (On a class of invariant sets of smooth dynamical systems). *Proc.* 5^{th} *Int. Conf. on Nonl. Oscill., Kiev*, 2:39–45, 1970.

[3] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Springer–Verlag, Berlin–Heidelberg, 1975. LN in Math. 470.

[4] R. Easton. Chain transitivity and the domain of influence of an invariant set. *In: The structure of attractors in dynamical systems, North Dakota State Univ, June 1977, Springer LN in Math. 668*, pages 95–102, 1978.

[5] A. Katok. Local properties of hyperbolic sets. *Appendix to the Russian translation of: Z. Nitecki, Differentiable Dynamics, 1971.*

[6] G. Kronin. Perturbations of Mather and Sacker–Sell spectra (to appear).

[7] Z. Nitecki. *Differentiable Dynamics.* MIT Press, Boston, 1971.

[8] S. Yu. Pilyugin. *The space of dynamical systems with the* C^0*–topology.* Springer–Verlag, Berlin–Heidelberg, 1994. LN in Math. 1571.

[9] V. A. Pliss. *Интегральные множества периодических систем дифференциальных уравнений (Integral sets of periodic systems of differential equations).* Nauka, Moscow, 1977.

[10] R. J. Sacker and G. R. Sell. A spectral theory for linear differential systems. *J. Diff. Equations*, 27:320–358, 1978.

[11] M. Shub. *Global stability of dynamical systems.* Springer-Verlag, New York, 1987.