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INCOMPLETE CORRECTIONS IN NONLINEAR PROBLEMS

T. EIROLA, A.M. KRASNOSEL'SKII, M.A. KRASNOSEL'SKII, N.A. KUZNETSOV, AND O. NEVANLINNA

ABSTRACT. A general setting of incomplete iterations:

$$u_{n+1}(x) = u_n(x) + \lambda_n(x)[Fu_n(x) - u_n(x)]$$

is considered. Using methods from the theory of positive operators general results about convergence of this are given. These results relate the convergence of the iterations to the series $\sum_n \lambda(x)$ and it's divergence. As an application we consider asynchronous iteration with parallel processors.

1. INTRODUCTION

Let Ω be a set and $\mathcal{E} = \mathcal{E}(\Omega)$ – some space of functions from Ω to a Banach space $\mathcal{B}$. In this work we consider (in general case nonlinear) operators $F : \mathcal{E} \to \mathcal{E}$ and study the convergence of sequences given by iteration

$$u_{n+1}(x) = u_n(x) + \lambda_n(x)[Fu_n(x) - u_n(x)]$$

for $n = 0, 1, 2, \ldots$ with some initial function u_0.

The results of this work can be interpreted in two ways.

On one hand, they give conditions for stability of equilibria for new classes of nonlinear systems. These classes generalize the finite dimensional desynchronized systems (see [1], [2],[3],[4],[5], [6],[7], and [10] and references there). They may be useful as models of systems where corrections of the state have to satisfy certain restrictions (ecological, economical, and some control problems in technology).

On the other hand, the results of this work can be used as conditions for applicability of a new class of procedures to construct approximate solutions for nonlinear operator equations $u = Fu$. The new class of approximate procedures – incomplete iterations – is especially interesting when the computations are carried out using multiprocessor computers or computer networks asynchronously. This class generalizes different variants of the under-relaxation method. The present theorems can also be thought as assertions about stability or robustness of Seidel-Nekrasov -type numerical methods.

Some of the present results were announced in [8]. It seems that some of these results have been unnoticed even in finite dimensional linear cases.

A special property of the iterative method, which is proposed and studied here, is the possibility to take the next step when knowing only some of the components of the residual. This is important for example in the applications to control problems and in construction of ecological models.

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2. Problem formulation

Let $\mathcal{B}$ be a Banach space with norm $|\cdot|$, Ω a compact subset of a finite dimensional Euclidean space and μ a (not necessarily continuous) positive measure in Ω with $\mu\Omega < \infty$. Denote by $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{B})$ one of the following spaces of $\mathcal{B}$–valued functions: the space $\mathcal{C} = \mathcal{C}(\Omega, \mathcal{B})$ of continuous functions in Ω with the uniform norm, the space $M = M(\Omega, \mathcal{B})$ of bounded functions in Ω with the norm $\|u\|_M = \sup_{x\in\Omega} |u(x)|$, and $L_p = L_p(\Omega, \mathcal{B})$ $(1 \le p \le \infty)$ w.r.t measure μ with the usual norm. In the latter case ($\mathcal{E} = L_p$) we assume for simplicity that $\mathcal{B}$ is finite dimensional.

If Ω is finite ($= \{1, \dots, p\}$) we can also consider the space $\mathcal{E} = E_1 \times \cdots \times E_p$, where the spaces E_i may be different for different i's.

Let $\mathcal{E}_0$ denote the corresponding (to $\mathcal{E}$) space of real–valued functions and denote by K_+ the cone of nonnegative functions in $\mathcal{E}_0$. Denote $w \succeq v$ if $w - v \in K_+$. For $u \in \mathcal{E}$ denote by $|u| \in K_+$ the function $|u|(x) := |u(x)|$. Unless explicitly denoted we will use $\|\cdot\|$ both in $\mathcal{E}$ and $\mathcal{E}_0$. Thus for $u \in \mathcal{E}$ we have $\|\,|u|\,\| = \|u\|$.

Consider a nonlinear operator F on $\mathcal{E}$ satisfying a generalized Lipschitz condition

$$|Fu - Fv| \preceq A|u - v| \qquad (u, v \in \mathcal{E}) \,, \tag{2.1}$$

where A is a linear operator in $\mathcal{E}_0$, which is positive with respect to K_+, i.e., $AK_+ \subset K_+$. The positivity means that if $u \succeq 0$ then also $Au \succeq 0$

If $\mathcal{E}$ is one of the spaces L_p, then the inequality in (2.1) is to be satisfied with almost all $x \in \Omega$, and K_+ is the cone of functions which are nonnegative almost everywhere.

When Ω consists of finite number of points, then $\mathcal{E}_0$ is finite dimensional and A is a matrix with nonnegative entries.

Let Ω be a domain in a finite dimensional space. Then an important example of operators F and A are integral operators with nonnegative kernels

$$Fu(x) = \int_\Omega K[x, s, u(s)]ds$$
$$Au(x) = \int_\Omega k(x, s)u(s)ds.$$

Any positive linear operator in L_p, $\mathcal{C}$ or M is bounded (see [9]), hence the spectral radius $\rho = \rho(A; \mathcal{E}_0)$ of A is finite. In the following it will not be assumed that F or A would be completely continuous, neither that A would have eigenfunctions.

If

$$\rho(A; \mathcal{E}_0) < 1 \,, \tag{2.2}$$

then with respect to some norm $\|\cdot\|_A$ in $\mathcal{E}_0$ (see e.g. [9]) A is a contraction. Then F is also a contraction and the nonlinear equation

$$u(x) = Fu(x) \tag{2.3}$$

has a unique solution u_*. This is the limit of successive approximations $u_{n+1}(x) = Fu_n(x)$ or, equivalently,

$$u_{n+1}(x) = u_n(x) + [Fu_n - u_n](x) \,. \tag{2.4}$$

The sequence (2.4) converges for any initial approximation $u_0 \in \mathcal{E}$.

Condition (2.2) is the essential assumption of this article. There exists a considerable amount of literature (see e.g. [9] and refences there) considering methods to check this condition using value of A at just one element of K_+.

Consider the sequence

$$u_{n+1}(x) = u_n(x) + \lambda_n(x)[Fu_n - u_n](x) \ , \tag{2.5}$$

where the λ_n's are measurable (and in the case of $\mathcal{E} = \mathcal{C}$ also continuous) functions. It will always be assumed that

$$0 \leq \lambda_n(x) \leq 1 \tag{2.6}$$

for all $x \in \Omega$ and $n \geq 0$. About the functions λ_n we might know only that they belong to certain sets (for example that every λ_n is a charcteristic function of a subset of Ω) or some information about the series

$$\lambda_0(x) + \lambda_1(x) + \cdots + \lambda_n(x) + \dots \tag{2.7}$$

Equation (2.5) can be seen as a process of constructing approximations to the solution u_* of equation (2.3).

Scheme (2.5) covers situations where the explicit form of operator F is not known; for corrections $\lambda_n(x)[Fu_n - u_n](x)$ only approximate information about u_n or values in a subset of Ω may be sufficient.

We are interested in conditions for convergence in some sense to u_* in different spaces $\mathcal{E}$. These conditions are stability conditions of equilibrium u_* of the system whose state is a function $u \in \mathcal{E}$ and dynamics is described by equation (2.5).

3. The convergence theorem

The convergence of (2.5) will essentially depend on the divergence of the series (2.7). We will consider three cases:

(a) Uniform divergence, i.e. for all $M > 0$ there exist N_M such that

$$\sum_{j=0}^{N_M} \lambda_j(x) \geq M \tag{3.1}$$

for all $x \in \Omega$,

(b) Arithmetic divergence, i.e. there exist $N \geq 1$ and $\beta > 0$ such that for all $x \in \Omega$, $k = 0, 1, 2, \dots$

$$\sum_{j=kN}^{(k+1)N-1} \lambda_j(x) \geq \beta \ , \tag{3.2}$$

(c) Divergence almost everywhere.

In the case $\mathcal{E} = L_p$ the inequalities (3.1) and (3.2) are required only almost everywhere.

Theorem 3.1. *Suppose F satisfies (2.1), where A is a positive operator such that $\rho := (A; \mathcal{E}_0) < 1$.*

A *Assume the series (2.7) diverges uniformly. Then the sequence (2.5) converges to u_* in the norm of $\mathcal{E}$. Further, if the divergence is arithmetic, then the convergence is geometric, i.e.*

$$\|u_n - u_*\| \leq C(u_0)q^n, \tag{3.3}$$

where $C(u_0)$ is a positive constant (the dependence of which on u_0 will be discussed below,) and

$$q = [\rho + (1-\rho)\exp(-\beta)]^{1/N}. \tag{3.4}$$

B *Assume $\mathcal{E} = M$ and that A maps also $L_p \to L_p$ for some $p < \infty$. If the series (2.7) diverges almost everywhere, then the sequence (2.5) converges almost everywhere.*

If in the situation of Theorem 3.1 every function λ_n in series (2.7) is a characteristic function of a subset $G_n \subset \Omega$, then condition (3.2) naturally takes the form

$$\bigcup_{j=kN}^{(k+1)N-1} G_j = \Omega \; . \tag{3.5}$$

Then (3.3) and (3.4) hold with $\beta = 1$. In this case it is natural to assume $N \geq 2$.

Before the proof of Theorem 3.1 we prove some simple lemmas.

Consider a sequence of functions defined recursively from $z_0 \in \mathcal{E}_0$:

$$z_{n+1}(x) = [1 - \lambda_n(x)]z_n(x) + \lambda_n(x)Az_n(x). \tag{3.6}$$

Lemma 3.2. *Assume* (2.1) *and* (2.6) *hold together with*

$$|u_0 - u_*| \preceq z_0. \tag{3.7}$$

Then

$$|u_n - u_*| \preceq z_n. \tag{3.8}$$

Proof. Consider the sequence $y_n := u_n - u_*$. As

$$y_{n+1}(x) = [1 - \lambda_n(x)]y_n(x) + \lambda_n(x)[Fu_n(x) - u_*(x)]$$

and $u_* = Fu_*$ we get

$$|y_{n+1}(x)| \leq [1 - \lambda_n(x)]|y_n(x)| + \lambda_n(x)|Fu_n(x) - Fu_*(x)|.$$

Thus from (2.1)

$$|y_{n+1}(x)| \leq [1 - \lambda_n(x)]|y_n(x)| + \lambda_n(x)A|y_n(x)|. \tag{3.9}$$

Then by induction: assume (3.8) holds for n, i.e., $|y_n| \preceq z_n$. Then by (3.9),

$$\begin{aligned} |y_{n+1}(x)| &\leq [1 - \lambda_n(x)]|y_n(x)| + \lambda_n(x)A|y_n(x)| \leq \\ &\leq [1 - \lambda_n(x)]z_n(x) + \lambda_n(x)Az_n(x) = z_{n+1}(x). \end{aligned}$$

Naturally, in the case $\mathcal{E} = L_p$ the inequalities above hold only for almost every $x \in \Omega$. $\square$

The following lemma is related to general theory of linear positive operators.

Lemma 3.3. *Let E be a Banach space with a cone K. Let A be a linear operator, which is positive with repect to K ($AK \subset K$) and $\rho(A)$ the spectral radius of A. Then for every $\varepsilon > 0$ and $f \in K$ there exists a $g \succeq f$ (i.e. $g - f \in K$) such that*

$$Ag \preceq [\rho(A) + \varepsilon]g \; . \tag{3.10}$$

Proof. The element g can be defined by formula

$$g = (I - A_\varepsilon)^{-1} f \quad , \qquad A_\varepsilon = [\rho(A) + \varepsilon]^{-1} A \tag{3.11}$$

The defifnition of the operator A_ε implies

$$\rho(A_\varepsilon) = \rho(A)[\rho(A) + \varepsilon] < 1.$$

Therefore the operator $(I - A_\varepsilon)^{-1}$ exists. From the first relation of (3.11) the equality

$$g - [\rho(A) + \varepsilon]^{-1} Ag = f$$

follows. But

$$g = f + A_\varepsilon f + A_\varepsilon^2 + \cdots + A_\varepsilon^n + \ldots$$

and the positivity of A_ε guarantees $g \succeq 0$ and furthermore $Ag \succeq 0$. Thus $g \succeq f$ and (3.10) hols. □

Lemma 3.4. *If $Az_0 \preceq z_0$ and (2.6) holds. Then the sequence (3.6) satisfies*

$$Az_n \preceq z_{n+1} \preceq z_n \ . \tag{3.12}$$

Proof. By induction. Case $n = 0$:

$$z_1(x) = [1 - \lambda_0(x)]z_0(x) + \lambda_0(x)Az_0(x) \le [1 - \lambda_0(x)]z_0(x) + \lambda_0(x)z_0(x) = z_0(x)$$

$$Az_0(x) = z_1(x) - [1 - \lambda_0(x)][z_0(x) - Az_0(x)] \le z_1(x) \ .$$

Then the induction hypotheses implies $Az_{n+1} \preceq Az_n \preceq z_{n+1}$, from which as above

$$z_{n+2}(x) = [1 - \lambda_{n+1}(x)]z_{n+1}(x) + \lambda_{n+1}(x)Az_{n+1}(x) \le z_{n+1}(x)$$

and

$$Az_{n+1}(x) = z_{n+2}(x) - [1 - \lambda_{n+1}(x)][z_{n+1}(x) - Az_{n+1}(x)] \le z_{n+2}(x) \ .$$

□

Lemma 3.5. *In the situation of Lemma 3.4 assume $e \in K_+$ and $\nu \in (0, 1)$ are such that $Ae \preceq \nu e$. Then for the sequence (3.6) holds:*

$$\text{if} \quad z_m \preceq c_m e \quad \text{then} \quad z_{m+k}(x) \le c_m\{\nu + (1 - \nu)\prod_{j=1}^{k}[1 - \lambda_{m+j-1}(x)]\}e(x). \tag{3.13}$$

Proof. By induction. Case $k = 0$ is trivial. Then using $z_{m+k} \preceq z_m$

$$\begin{aligned} z_{m+k+1}(x) &= [1 - \lambda_{m+k}(x)]z_{m+k}(x) + \lambda_{m+k}(x)Az_{m+k}(x) \le \\ &\le c_m[1 - \lambda_{m+k}(x)]\{\nu + (1 - \nu)\prod_{j=1}^{k}[1 - \lambda_{m+j-1}(x)]\}e(x) + \\ &\quad + \lambda_{m+k}(x)c_m Ae(x) \le \\ &\le c_m\{\nu + (1 - \nu)\prod_{j=1}^{k+1}[1 - \lambda_{m+j-1}(x)]\}e(x) \ . \end{aligned}$$

□

Let us turn to the proof of Theorem 3.1:

Proof. Take a function $v \in K_+$ s.t. $\inf_x v(x) > 0$ and the function $v - |u_0 - u_*|$ is in K_+. For $\varepsilon \in (0, 1 - \rho)$ use Lemma 3.3 to get $e \in K_+$ such that $v \preceq e$ and $Ae \preceq (\rho + \varepsilon)e$. Take $z_0 = e$ and apply Lemma 3.2. Then it suffices to prove the corresponding statements about converging to zero of the sequence $\{z_n\}_{n\ge0}$ in $\mathcal{E}_0$. Note that the assertions of Lemmas 3.4 and 3.5 are now valid with $\nu := \rho + \varepsilon$.

A If (2.7) diverges uniformly, then for any $\beta > 0$ there exists an increasing sequence $\{d_k\}_{k\ge0}$ of integers such that

$$\sum_{j=d_k}^{d_{k+1}-1} \lambda_j(x) \ge \beta \ . \tag{3.14}$$

Indeed, take $d_0 = 0$, $d_{k+1} = N_{d_k+\beta} + 1$.

Denote $\Delta_k := d_{k+1} - d_k$. Using (3.13), the inequality between arithmetic and geometric meanvalues, and (3.14) one gets for $\gamma_k := \inf\{c > 0; z_{d_k} \preceq ce\}$:

$$\gamma_{k+1} \leq \gamma_k \sup_{x\in\Omega}\{\nu + (1-\nu)\prod_{j=1}^{\Delta_k}[1 - \lambda_{d_k+j-1}(x)]\} \leq$$

$$\leq \gamma_k \sup_{x\in\Omega}\{\nu + (1-\nu)\left(\frac{1}{\Delta_k}\sum_{j=1}^{\Delta_k}[1-\lambda_{d_k+j-1}(x)]\right)^{\Delta_k}\} \leq$$

$$\leq \gamma_k\{\nu + (1-\nu)[1 - \frac{\beta}{\Delta_k}]^{\Delta_k}\} \leq \gamma_k\{\nu + (1-\nu)e^{-\beta}\} .$$

Thus $\gamma_k \leq \mu^k$, where $\mu = \rho+\varepsilon+(1-\rho-\varepsilon)e^{-\beta} < 1$, $\|u_{d_k} - u^*\| \leq \mu^k\|e\|$, and $\|u_n - u^*\| \to 0$ by monotonicity of $\{z_n\}_{n\geq 0}$.

If, further, (3.2) holds, then one can take $\Delta_k = N$ for all k and get as above

$$\gamma_k \leq \{\rho + \varepsilon + (1-\rho-\varepsilon)[1 - \frac{\beta}{N}]^N\}^k . \tag{3.15}$$

Because $[1 - \frac{\beta}{N}]^N < e^{-\beta}$ we could have taken $\varepsilon \in (0, 1-\rho)$ so small that

$$\rho + \varepsilon + (1-\rho-\varepsilon)[1 - \frac{\beta}{N}]^N \leq \rho + (1-\rho)e^{-\beta} , \tag{3.16}$$

which implies (3.4).

B Assume (2.7) diverges for almost every $x \in \Omega$. We will show that $z_n(x) \to 0$ for a.e. $x \in \Omega$. By monotonicity (Lemma 3.4) the sequence $\{z_n(x)\}_{n\geq 0}$ is nonincreasing, thus there exists a pointwise limit $z_* \in L_p \cap M$.

Suppose the function z_* is positive on a set of positive measure. Let α_* be the the infimum of positive α's satisfying

$$\mu\{x \ : \ z_*(x) \geq \alpha e(x)\} = 0 . \tag{3.17}$$

This α_* is positive. Fix positive numbers α_1, α_2, and β satisfying ($\nu = \rho + \varepsilon < 1$)

$$\nu\alpha_1 + \beta < \alpha_2 < \alpha_* < \alpha_1. \tag{3.18}$$

Consider the set

$$G_* := \{x \in \Omega \ : \ z_*(x) > \alpha_2 e(x)\} . \tag{3.19}$$

Then by the definition of α_* we have $\mu G_* > 0$. Denote

$$m := \inf_{x\in\Omega} e(x) \qquad , \qquad M := \sup_{x\in\Omega} e(x).$$

and set

$$\delta := \frac{\mu G_*}{2}\left(\frac{m\beta}{M\|A\|_{L_p\to L_p}}\right)^p . \tag{3.20}$$

Since the sequence $z_n(x)$ is nonincreasing, converges to $z_*(x)$, and almost everywhere

$$\alpha_1 e(x) - z_*(x) \geq (\alpha_1 - \alpha_*)e(x) \geq m(\alpha_1 - \alpha_*) > 0 ,$$

it is possible to take an n_0 such that

$$\mu\{x \in \Omega \ : \ z_{n_0}(x) \leq \alpha_1 e(x)\} > \mu\Omega - \delta . \tag{3.21}$$

Set

$$G_1 := \{x \in \Omega \ : \ z_{n_0}(x) > \alpha_1 e(x)\} \ .$$

From (3.21) it follows that $\mu G_1 < \delta$. Then z_{n_0} satisfies

$$z_{n_0}(x) \le \begin{cases} \alpha_1 e(x), & \text{if } x \in \Omega \setminus G_1 \\ M & \text{if } x \in G_1 \end{cases} \ . \tag{3.22}$$

It follows from (3.22), definition (3.19) of G_*, and the monotonicity of the sequence $z_n(x)$ that for every $n \ge n_0$ holds

$$z_n(x) \le \begin{cases} \alpha_1 e(x), & \text{if } x \in \Omega \setminus G_1 \ , \\ M & \text{if } x \in G_1 \end{cases} \tag{3.23}$$

and

$$z_n(x) \ge \alpha_2 e(x) \quad \text{ for } x \in G_* \ . \tag{3.24}$$

Let χ be the characteristic function of G_1. From (3.23) it follows that for every $x \in \Omega$, $n \ge n_0$

$$A z_n(x) \le A[\alpha_1 e + M\chi](x) \le \alpha_1 \nu e(x) + MA\chi(x). \tag{3.25}$$

Since

$$\|\chi\|_{L_p} = (\mu G_1)^{1/p} < \delta^{1/p} \ ,$$

and because of (3.20) we have

$$\|MA\chi\|_{L_p} \le M\|A\|_{L_p \to L_p} (\mu G_1)^{1/p} \le \left[\frac{\mu G_*}{2}\right]^{1/p} m\beta \ . \tag{3.26}$$

Consider the set

$$G_2 := \{x \in \Omega \ : \ |MA\chi(x)| \ge m\beta\}. \tag{3.27}$$

According to the Chebyshev inequality

$$\mu G_2 \le \frac{\|MA\chi\|_{L_p}^p}{m^p \beta^p} \ ,$$

and therefore(3.26) implies $\mu G_2 \le \frac{1}{2}\mu G_*$, and since $\mu G_* > 0$ we also have $\mu(G_* \setminus G_2) > 0$. From (3.25) and (3.27) it follows that for every $x \in G_* \setminus G_2$ and $n \ge n_0$

$$A z_n(x) \le \alpha_1 \nu e(x) + m\beta \le (\alpha_1 \nu + \beta) e(x) \ ,$$

therefore using (3.24) for every $x \in G_* \setminus G_2$ and $n \ge n_0$ holds

$$z_n(x) - A z_n(x) \ge (\alpha_2 - \alpha_1 \nu - \beta) e(x) \ge (\alpha_2 - \alpha_1 \nu - \beta) m = \text{const} > 0 \ . \tag{3.28}$$

Inequality (3.28) is the basic part for proving contradiction in the assumption of the positivity of z_* on a set of positive measure.

Definition (3.6) of sequence z_n implies identity

$$z_{n_0+k+1}(x) = z_{n_0}(x) - \sum_{j=0}^{k} \lambda_{n_0+j}(x)[z_{n_0+j}(x) - A z_{n_0+j}(x)] \ .$$

Therefore (3.28) implies for $x \in G_* \setminus G_2$ and $n \ge n_0$

$$z_{n_0+k+1}(x) \le z_{n_0}(x) - (\alpha_2 - \alpha_1 \nu - \beta) m \sum_{j=0}^{k} \lambda_{n_0+j}(x) \ ,$$

and since $z_{n_0}(x) - z_{n_0+k}(x) \le M$ we have for $x \in G_* \setminus G_2$ and $k = 1, 2, \dots$

$$\sum_{j=0}^{k} \lambda_{n_0+j}(x) \le \frac{M}{(\alpha_2 - \alpha_1\nu - \beta)m} < \infty \ .$$

The last inequality implies the convergence of the series (2.7) for values of x in a set of positive measure, which is a contradiction to the assumptions of the theorem.

This finishes the proof of Theorem 3.1. $\square$

Remark 3.1. One gets easily from (3.4), (3.15), and (3.16) that

$$c(u_0) = q^{-N}\|e\|$$

is good for (3.3). In the case $\mathcal{E} = L_p$ this is uniform for all u_0 s.t. $|u_0 - u_*| \preceq v$.

Inspection of the beginning of the proof of Theorem 3.1 shows that in cases $\mathcal{E} = \mathcal{C}$ and $\mathcal{E} = M$ we can first take e_0 such that $e_0(x) \ge 1$ and $Ae_0 \preceq (\rho + \varepsilon)e_0$ and then set $e := \|u_0 - u_*\|e_0$. This gives

$$c(u_0) = q^{-N}\|u_0 - u_*\|\|e_0\|$$

in these cases.

4. Asynchronous iteration

As an application of the theory we will consider the iteration

$$w^{n+1} = f(w^n) \tag{4.1}$$

for solving equation $w = f(w)$ in a space E, which is a product of p normed spaces: $E = E_1 \times \cdots \times E_p$. Divide w into components $w(1), \dots, w(p)$. Then we can take p processors and let processor j compute

$$w_{n+1}(j) = f_j(w_n) \ .$$

If we make this in a synchronized way so that the processors that have done their jobs wait until others are ready, then we are back in the original iteration (4.1). Suppose, however, that at time t, when processor j gets ready we give it immediately the task to compute $f_j(w^t)$, where w^t consists of the latest updates of the components at time t. Then we are in a situation where the updates are computed from different values of w asynchronously.

This can be modelled in the following way. Denote $u(\cdot, j) := [u(1, j), \dots, u(p, j)]$ the value for w that processor j holds in its memory. When j gets ready $u(\cdot, j)$ will be updated $u(j, j) \leftarrow f_j(u(\cdot, j))$ and other $u(i, j)$'s by the newest versions of them, i.e. by $u(i, 0)$'s, where $u(\cdot, 0)$ is the collection of current updates held in the main memory. This can be cast into the form of (2.5) by setting $\Omega := \{(i, j)\}_{j=0,\dots,p}^{i=1,\dots,p}$, $\mathcal{E} := E^{p+1}$. Then n is increased every time when one of the processors gets ready. Set

$$u_{n+1}(i, j) = [1 - \lambda_n(i, j)]u_n(i, j) + \lambda_n(i, j)Fu_n(i, j) \ , \tag{4.2}$$

where

$$Fu(i, j) = \begin{cases} f_i(u(\cdot, i)), & \text{if } i = j \ne 0 \text{ or } j = 0 \\ u(i, 0), & \text{otherwise.} \end{cases} \tag{4.3}$$

When processor j gets ready set $\lambda_n(j, j) = \lambda_n(j, 0) = 1$ and all other $\lambda_n(i, k)$'s to zero.

Now we can apply Theorem 3.1 for this iteration. Recall that

$$|v| := [\|v(1)\|, \dots, \|v(p)\|]^T \ .$$

Theorem 4.1. *Suppose f satisfies*

$$|f(v) - f(w)| \preceq A|v - w| \qquad (v, w \in E) \ , \tag{4.4}$$

where A is a nonnegative $p \times p$ matrix such that $\rho(A) < 1$. Assume every processor always can perform its task in a finite time. Then the sequence $\{u_n\}_{n\geq 0}$ defined by (4.2) converges to the ($p+1$ –fold copy of the) unique fixed point of f. If there is an upper bound for the times required by the processors, then the convergence is geometric.

Proof. We have to show that our assumptions on f imply the conditions on F assumed in Theorem 3.1. Let J_k be the diagonal matrix having zero at the k'th diagonal entry and ones elsewhere, and A_k the matrix consisting of zeroes except the k'th row, which equals that of A, i.e. $A_k = (I - J_k)A$. Set

$$\tilde{A} := \begin{bmatrix} 0 & A_1 & A_2 & \dots & A_p \\ J_1 & A_1 & 0 & \dots & 0 \\ J_2 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_p & 0 & 0 & \dots & A_p \end{bmatrix} . \tag{4.5}$$

Then it is easy to see that (4.4) implies (2.1) with $\tilde{A}$ in the place of A. Then the proof is completed using the lemma below. ☐

Lemma 4.2. *Let A be a nonnegative square matrix with $\rho(A) \leq 1$ and $\tilde{A}$ constructed from A as above. Then $\rho(\tilde{A}) \leq \sqrt{\rho(A)}$.*

Proof. Assume first that A is positive (then the general case follows via a limit procedure). Take a positive vector x such that $Ax = \rho x$ $(= \rho(A)x)$ and set $\tilde{x} := [\rho x^T, x^T, \dots, x^T]^T \in \mathbf{R}^{(p+1)p}$. Then

$$\tilde{A}^2\tilde{x} = \tilde{A} \begin{bmatrix} \rho x \\ \rho x \\ \vdots \\ \rho x \end{bmatrix} = \rho \begin{bmatrix} \rho x \\ J_1 x + \rho(I - J_1)x \\ J_2 x + \rho(I - J_2)x \\ \vdots \\ J_p x + \rho(I - J_p)x \end{bmatrix} \preceq \rho \begin{bmatrix} \rho x \\ x \\ \vdots \\ x \end{bmatrix} = \rho\tilde{x} \ .$$

Since $\tilde{A}^2$ is nonnegative and $\tilde{x}$ is positive we get (e.g. by Lemma 9.4 of [9]) that

$$\rho(\tilde{A}^2) \leq \rho(A) \ .$$

☐

Application to Picard-Lindelöf –iteration

Our aim here is to show that asynchronous iteration of dynamical processes can also be treated within this formalism.

The technique goes under the name *waveform relaxation* in the context of electrical networks. Since Picard and Lindelöf used such an iteration to prove existence for initial value problems, the iteration is sometimes called Picard-Lindelöf –iteration.

For sake of simplicity we consider here the linear constant (real) coefficient case:

$$\dot{u}(t) + Lu(t) = g(t) \quad , \quad u(0) = b \ .$$

The idea is to split $L = M - N$ to obtain

$$\dot{u}(t) + Mu(t) = Nu(t) + g(t) \quad , \quad u(0) = b \ ,$$

i.e. $u = f(u)$, where

$$f(u)(t) = Ku(t) + \int_0^t e^{-(t-s)M} g(s) ds + e^{-tM} b \ ,$$
$$Ku(t) = \int_0^t e^{-(t-s)M} Nu(s) ds \ .$$

We will apply Theorem 4.1 to this situation. Since f is affine (4.4) becomes

$$|Ku| \preceq A|u| \qquad (u \in E) \ . \tag{4.6}$$

For parallel processing it is a natural to choose M to be block diagonal:

$$M = \mathrm{diag}[M_1, \ldots, M_p] \ .$$

Partition $E = E_1 \times \cdots \times E_p$ and $N = \{N_{i,j}\}$, respectively, with $N_{i,i} = 0$, $i = 1, \ldots, p$. Here E_j is the space of continuous functions $[0, T] \to \mathbf{R}^{n_j}$. Clearly, K can be thought of as an operator matrix $K = \{K_{i,j}\}$ (with $K_{i,i} = 0$).

Let a norm $\|\cdot\|$ be given in every E_j and let $\|K_{i,j}\|$ be the corresponding operator norm of $K_{i,j} : E_j \to E_i$. We put $A := \{\|K_{i,j}\|\}$. Then clearly $|Ku| \preceq A|u|$.

The focus is to define the norms in E_j's so that $\rho(A) < 1$.

Claim 4.3. *For any $\varepsilon > 0$ there exist norms in E_j's such that $\rho(A) < \varepsilon$.*

Proof. If $|\cdot|$ is any norm in $\mathbf{R}^{n_j}$ then put

$$\|u_j\|_\alpha := \max_{[0,T]} |e^{-\alpha t} u_j(t)| \ .$$

A straightforward calculation gives

$$\|K_{i,j}\|_\alpha \leq \frac{1}{\alpha - |M_i|} |N_{i,j}|$$

and in particular, $\rho(A) \leq \max_i \sum_j \|K_{i,j}\|_\alpha < \varepsilon$ for α large enough. $\square$

Above we have to take $\alpha \approx \max_i(|M_i| + \sum_j |N_{i,j}|)$ and this can be all too large in practice:

In order to know that $|u(T)|$ is small we must make $e^{\alpha T}\|u_j\|_\alpha$ small. To handle such a case we allow $|M_i|$ and $|N_{i,j}| \gg 1$ but assume instead the following estimates:

$$|e^{-tM_i}| \leq C_0 e^{-tm}$$

and

$$|M_i e^{-tM_i}| \leq C_1 / t \ ,$$

where C_0, C_1, and m are all of moderate size. For $\theta \in (0, 1)$ we then have

$$|M_i e^{-tM_i}| \leq \left(\frac{C_1}{t}\right)^{1-\theta} |M_i|^\theta C_0^\theta e^{-t\theta m} \ .$$

This immediately gives

$$\|K_{i,j}\| \leq \int_0^t |e^{-tM_i} N_{ij}| dt \leq$$
$$\leq \int_0^\infty |M_i e^{-tM_i}| dt |M_i^{-1} N_{i,j}| \leq C_\theta |M_i|^\theta |M_i^{-1} N_{i,j}| \ ,$$

where $C_\theta = O(\frac{1}{\theta})$ as $\theta \to 0$. For example, $C_\theta := C_0^\theta C_1^{1-\theta} \frac{1}{\theta} \frac{m+1}{m}$ will do. This leads to the estimate

$$\rho(A) \leq C_\theta \max_i |M_i|^\theta \sum_j |M_i^{-1} N_{i,j}| \ .$$

If this estimate is too large one can again use exponential weighting – but this time it is only needed in the scale of $|M_i|^\theta$.

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(T. Eirola and O. Nevanlinna) Institute of Mathematics, Helsinki University of Technology, Finland

(A.M. Krasnosel'skii, M.A. Krasnosel'skii, and N.A. Kuznetsov) Institute of Information Transmission Problems, Russian Academy of Sciences