J. of Comp. and Applied Mathematics **45**, 65-73 (1993).

ASPECTS OF BACKWARD ERROR ANALYSIS OF NUMERICAL ODE'S

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ABSTRACT. In this paper some preliminary cases are studied, to see how the idea of backward error analysis carries over to numerical solution of ordinary differential equations. The interest here is in one-step methods and infinite time intervals.

1. INTRODUCTION

The idea of the backward error analysis of a numerical method is the question: What does the method solve? This means that instead of studying the difference between the numerical solution and the true one we look for a nearby problem which would be solved (in some sense) exactly by the numerical result. If the problem is from physics this is also related to the question of whether the behaviour of the numerical solution is physically meaningful. This work is a preliminary study about how this idea can be used in the context of solving ordinary differential equations by standard one-step methods.

Truncation errors can cause large deviations from the true solution over long time intervals. In such situations the system is typically very sensitive and big deviations can be caused also by small perturbations in the system. The question here is whether these deviations are similar enough so that the former could be produced by the latter ones as well. Now the numerical solutions are discrete functions, and are hence unable to fully represent a solution of any differential equation. At most we may hope that the values of the numerical solution are equal to a sampled continuous solution of a nearby equation.

In this paper we restrict ourselves to truncation errors only, i.e. we assume that the computations are performed with exact arithmetic.

Consider an autonomous differential equation

$$\dot{x}(t) = f(x(t)) \tag{1.1}$$

in a compact subset Ω of $\mathbf{R}^n$. We assume f to be sufficiently smooth. Denote by ϕ_f the flow of f, i.e., if $x_0 \in \Omega$, then $x(t) = \phi_f(t, x_0)$ is the solution of (1.1) defined for all $t \in \mathbf{R}$ as long as it stays inside Ω. A one-step numerical method (which

1991 *Mathematics Subject Classification.* Primary 65L; Secondary 34C.
Key words and phrases. one-step methods, backward error analysis.

we assume to be fixed and to have order p) applied to (1.1) defines a mapping $\psi_h : \Omega \to \mathbf{R}^n$ — a local diffeomorphism for small enough stepsize h — such that

$$||\psi_h - \phi_f(h,\cdot)||_{\mathcal{C}^0} \leq Ch^{p+1} \quad . \tag{1.2}$$

We assume that the method is such that it commutes with differentiation w.r.t. the initial condition, i.e. $(\psi_h, \partial_x \psi_h)$ equals the map defined by the method when it is applied to system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) & x(0) &= x_0 \\ \dot{\Phi}(t) &= f_x(x(t))\Phi(t) & \Phi(0) &= I \quad . \end{aligned}$$

The common methods satisfy this assumption (see[4]). Then (by induction) (1.2) remains true if $\mathcal{C}^0$ is replaced by any $\mathcal{C}^k$ – provided f is smooth enough.

2. The problem

Let $x_0 \in \Omega$ and $x_j = \psi_h^j(x_0)$, $1 \leq j < N$. The question is: does there exist another $\mathcal{C}^k$−flow ϕ (flow: satisfying $\phi(t+\tau,\cdot) = \phi(t,\phi(\tau,\cdot))$) close to ϕ_f such that

$$\phi(jh, x_0) = x_j \quad 0 \leq j < N \quad ? \tag{2.1}$$

If N is finite it is not too difficult to make local perturbations to f such that this is satisfied. Thus we are interested in the case $N = \infty$, assuming the x_j's will stay in Ω . It is also possible to add some time dependent terms in (1.1) to get this satisfied, but that would take us out of the family of autonomous systems.

There is an obstacle to obtaining a positive answer in general:

Example 2.1. Take an f such that (1.1) has a nontrivial hyperbolic periodic trajectory Γ . Then (see [4]) the numerical method also has an invariant circle Γ^h $O(h^p)$ $\mathcal{C}^k$−close to Γ . We could find a flow ϕ for which Γ^h is a periodic trajectory, but then $\phi(h,\cdot)$ restricted to Γ^h will be conjugate to rotation of the unit circle S^1 by a fixed angle (i.e. there exists a homeomorphism $H : \Gamma^h \to S^1$ such that $\phi(h,\cdot) = H^{-1} \circ \mathit{rotation} \circ H$.) The restriction of ψ_h to Γ^h is just a diffeomorphism of Γ^h . Nothing indicates that it should be conjugate to rotation. It may have asymptotically stable and unstable periodic points on Γ^h . Over long time intervals these can cause an $O(h)$ difference to the behaviour of ψ_h compared to the behaviour of any flow on Γ^h . On the other hand note that e.g. in the case of exponentially stable Γ the behaviour of ψ_h is $O(h^p)$ close to that of $\phi_f(h,\cdot)$ when measured "in normal directions" (see Theorem 3.3).

To perturb the problem (1.1) we can change either the flow or the collection of initial points we like to consider. Having in mind the example above we can try to separate the drift along trajectories from other errors. There are also the choices of the time interval and the norm in which we measure the error. To compactify the rest of the text we put all these together and give the following definition:

Definition 2.1. Let f, Ω , and a (one-step) method ψ be given. For nonnegative $q_i, k, l,$ and M we say that

$$P(q_0, q_1, q_2, q_3, k, l, M)$$

holds in $U \subset \Omega$ if there exist $h_o > 0$ and constants C_i such that for every $h \in (0, h_o)$ we can find ϕ, η, τ satisfying

$$\sup_{0 \leq jh < M} ||\psi_h^j - \phi(\tau(\cdot, j), \eta(\cdot))||_{\mathcal{C}^k(\psi_h^{-j}(\Omega) \cap U)} \leq C_0 h^{q_0} \quad ,$$

where

$$\begin{array}{lll} \phi \text{ is a flow} & \text{with} & ||\phi - \phi_f||_{\mathcal{C}^k} \leq C_1 h^{q_1} \\ \eta \,:\, U \to \Omega & \text{with} & ||\eta - I||_{\mathcal{C}^k} \leq C_2 h^{q_2} \\ \tau \,:\, U \times \mathbf{Z} \to \mathbf{R} & \text{with} & ||\tau(\cdot, j) - jh||_{\mathcal{C}^l} \leq C_3 h^{q_3} \,. \end{array}$$

Thus ϕ, η, and τ correspond to pertubations in the system (ODE), the initial condition, and time scale, respectively. Here $||g||_{\mathcal{C}^k}$ means the supremum of $\sum_{j=0}^k |\partial_\zeta^j g(\zeta)|$ over the interior of the domain of g . $|\cdot|$ is the Euclidean length. We will write $q_i = \infty$ if we can put the corresponding C_i equal to zero. A q_i equal to zero will denote $o(1)$ in the place of the corresponding $C_i h^{q_i}$. Thus for $M > h > 0$ $P(\infty, \infty, \infty, \infty, 0, 0, M)$ implies $\psi_h = \phi_f(h, \cdot)$ in U . Assign 1 for "true" and 0 for "false". Then P is nonincreasing in any of its arguments.

3. Partial results and discussion

We assume that the method is fixed and f is sufficiently smooth and don't count, how many derivatives we need since this depends on the method. Standard textbooks on numerical solution of ODE's usually prove statements of the form

$$P(p, \infty, \infty, \infty, 0, 0, M) \tag{3.1}$$

in $\{x_0\}$ with some finite M. Such results are quite easily generalized to the statement $P(p, \infty, \infty, \infty, k, k, M)$ in any U such that $\phi_f(t, U) \subset \Omega$ for all $t \in [0, M]$ (see [4]). A decent backward error statement would be of the form

$$P(\infty, q, r, \infty, k, l, \infty) \tag{3.2}$$

If $r = \infty$ in (3.2) we say that the numerical map is *imbedded* in a nearby flow. As we saw in the periodic solution example above we cannot hope for this in general, because of the possible $O(h)$ time drift. Thus it seems that generally we have to be satisfied with at most

$$P(\infty, p, \infty, 1, k, k, \infty) \quad . \tag{3.3}$$

It is left as a challenge to the numerical ODE community to find cases where (3.3) holds. Here we give some special case results and expectations in that direction.

3.1. Linear systems. Any reasonable method produces linear ψ_h for linear f. We assume that the method is such. Then for a linear system in any bounded subset of $\mathbf{R}^n$ we have genuine imbedding $P(\infty, p, \infty, \infty, \infty, \infty, \infty)$. More precisely,

Proposition 3.1. *Assume f is linear: $f(x) = Ax$. Then for h small enough there exists a (linear) $g = f + O(h^p)$ s.t. $\psi_h = \phi_g(h, \cdot)$.*

Proof. Since $||\psi_h - \phi_f(h, \cdot)||_{\mathcal{C}^1} \leq Ch^{p+1}$ and $\phi_f(h, \cdot) = I + O(h)$, for h small enough the spectrum of ψ_h will be inside a small disc centered at 1 so that there exists a unique logarithm: B_h s.t. $\psi_h(x) = e^{hB_h}x$. Expanding:

$$I + hB_h + \frac{1}{2}(hB_h)^2 + \cdots = I + hA + \frac{1}{2}(hA)^2 + \cdots + O(h^{p+1}) \tag{3.4}$$

we get $B_h = A + O(h^p)$. Put $g(x) = B_h x$. $\square$

3.2. Around an equilibrium. We say that an equilibrium ($f(0) = 0$, say) is *hyperbolic* if $f'(0)$ has no purely imaginary eigenvalues. In a neighbourhood of such a point we get topological (i.e. $\mathcal{C}^0$) imbedding:

Proposition 3.2. *Let $f(0) = 0$ be hyperbolic. Then in a neighbourhood of 0, for h small there exists a topological (i.e. $\mathcal{C}^0$) flow $\tilde{\phi}$ $O(h^p)$ $\mathcal{C}^0$–close to ϕ_f s.t. $\psi_h = \tilde{\phi}(h, \cdot)$. In short, $P(\infty, p, \infty, \infty, 0, 0, \infty)$.*

Proof. For h small enough the fixed point 0 of ψ_h is hyperbolic ($\psi_h'(0)$ has no eigenvalues of modulus 1). By the Grobman–Hartman theorem (Theorem 7.1 for ψ_h and 7.2 for ϕ in [2]) in a neighbourhood of 0 there are unique homeomorphisms $\tilde{H}$ and H (indep. of t) such that

$$\psi_h \circ \tilde{H} = \tilde{H} \circ \psi_h'(0) \quad \text{and} \quad \phi_f(t, \cdot) \circ H = H \circ [\partial_x \phi_f(t, 0)]$$

As in the proof of Proposition 3.1 set $\psi_h'(0) = e^{hB_h}$ and define

$$\tilde{\phi}(t, x) := \tilde{H}\left(e^{tB_h}\tilde{H}^{-1}(x)\right) \quad .$$

Then $\tilde{\phi}$ is a flow and $\psi_h = \tilde{\phi}(h, \cdot)$.

It is easy to modify the proof of Theorem 7.1 in [2] to get that the map $\psi_h \in \mathcal{C}^1 \to \tilde{H} \in \mathcal{C}^0$ is locally Lipschitz. This gives the $O(h^p)$ $\mathcal{C}^0$–closeness. $\square$

What about differentiable imbedding? Let $\{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of $f'(0)$. A *resonance* of order $k \geq 2$ is an equation of the form $\sum_{i=1}^n \gamma_i \lambda_i = \lambda_j$, where the γ_i's are nonnegative integers such that $\sum_{i=1}^n \gamma_i = k$. If in Proposition 3.2 we assume further that there are no resonances of order $\leq k$. Then by the *smooth linearization* due to Sternberg and Chen (see [6]) $\tilde{H}$ –and consequently $\tilde{\phi}$ – is in $\mathcal{C}^\kappa$, where κ increases to infinity with k. This gives us $P(\infty, 0, \infty, \infty, \kappa, \kappa, \infty)$ in a neighbourhood of 0. The proof for the smooth linearization is quite tedious and it is not clear whether we can get the ($\mathcal{C}^\kappa$–) Lischitz continuity of $\tilde{H}$ and that way the $O(h^p)$–closeness here in order to obtain $P(\infty, p, \infty, \infty, \kappa, \kappa, \infty)$.

3.3. Around a periodic trajectory. In Example 2.1 above, i.e., in a neighbourhood of a hyperbolic periodic solution it is not known, whether (3.3) holds. In that direction we have following theorem, study of which started with discussions with Andrew Stuart, Bath University ([9]).

Theorem 3.3. *Let Γ^0 be an exponentially stable periodic trajectory of* (1.1) *and K a compact neighbourhood attracted by Γ^0. Then there exists $C > 0$ such that for h small enough there exist $\tilde{f}$ defined in K and $O(h^p)$ $\mathcal{C}^k$–close to f and $\tau : K \times \mathbf{Z} \to \mathbf{R}$ with $||\tau(\cdot, j) - jh||_{\mathcal{C}^0} \leq Ch$ satisfying*

$$\sup_{j \geq 0} ||\psi_h^j - \phi_{\tilde{f}}(\tau(\cdot, j), \cdot)||_{\mathcal{C}^k} \leq Ch^p \quad .$$

In short, $P(p, p, \infty, 1, k, 0, \infty)$.

Proof. Take first $k = 0$. In [3] and [4] it is shown that for h small enough ψ_h has a unique $\mathcal{C}^1$–circle Γ^h $O(h^p)$ $\mathcal{C}^1$–close to Γ^0. Denote by $U(\delta, \Gamma^0)$ the set of points within distance $\delta > 0$ from Γ^0. For any $x \in U(\delta, \Gamma^0)$ let $x^0 \in \Gamma^0$ and $x^h \in \Gamma^h$ be the points closest to x such that

$$< x - x^0, f(x^0) >= 0 \quad \text{and} \quad < x - x^h, f(x^0) >= 0 \quad ,$$

i.e. x^0 and x^h are the intersections of the plane perpedicular to Γ^0 with Γ^0 and Γ^h, respectively. Let $v(x)$ be the unit tangent to Γ^h at x^h (in the direction of the flow). Set $d(x) := x^h - x^0$ and

$$g_h(x) := f(x - d(x)) + < v(x), f(x^0) > v(x) - f(x^0) \quad .$$

Then g_h is $O(h^p)$ $\mathcal{C}^1$–close to f and at $x^h \in \Gamma^h$

$$g_h(x^h) =< v(x^h), f(x^0) > v(x^h) \quad ,$$

i.e. $g_h(x^h)$ is tangent to Γ^h at x^h. Take a $\mathcal{C}^\infty$ function η which is 1 in $U(\delta/2, \Gamma^0)$ and 0 outside $U(\delta, \Gamma^0)$. Set

$$\tilde{f}(x) := e\,[\eta(x) g_h(x) + (1 - \eta(x)) f(x)] \quad ,$$

where $e = 1 + O(h^p) \in \mathbf{R}$ is to be defined later. Then $\tilde{f}$ is $O(h^p)$ $\mathcal{C}^1$– close to f and has Γ^h as its exponentially stable periodic trajectory.

Let $\hat{\phi}$ be the flow of g_h and $\hat{T}$ the period of $\hat{\phi}_{|\Gamma^h}$. Let ρ be the rotation number of $\psi_{h|\Gamma^h}$ and set $e := \frac{\hat{T}\rho}{h}$, i.e. s.t. the rotation number of $\phi_{\tilde{f}}(h, \cdot)_{|\Gamma^h}$ equals ρ.

Rotation number is generally not a Lipschitz –continuous function even in the space of diffeomorphisms (see e.g. section 6.2 in [5]), but if H is a homeomorphism of the unit circle ε ($\mathcal{C}^0$–) close to rotation : $t \to t + \alpha \mod 1$, then $|\rho(H) - \alpha| \leq \varepsilon$. Indeed,

$$\rho(H) = \lim_{m \to \infty} \frac{\bar{H}^m(0)}{m} \underset{(\geq)}{\leq} \lim_{m \to \infty} \frac{m\alpha \overset{+}{(-)} m\varepsilon}{m} \quad .$$

Here $\bar{H}$ is the lift of H (see [1]).

Lemma 3.4. $e = 1 + O(h^p)$.

Proof. Mapping Γ^h by a suitable diffeomorphism to the unit circle $\hat{\phi}(h,\cdot)_{|\Gamma^h}$ acts like rotation $t \to t + h/\hat{T} \mod 1$ and $\psi_{h|\Gamma^h}$ like an $O(h^{p+1})$ perturbation of this. Hence $\rho(\psi_{h|\Gamma^h}) = h/\hat{T} + O(h^{p+1})$. □

Fix an $\omega^* \in \Gamma^h$ and set $\omega(\tau) = \phi_{\tilde{f}}(\tau, \omega^*)$ and

$$V_\delta := \bigcup_{|\tau| \le \delta} \tilde{W}^s_{\omega(\tau)} \cap U(\delta, \Gamma^0) \quad ,$$

where $\tilde{W}^s_\omega$ is the stable manifold of ω , i.e. the set of points x for which $\lim_{t\to\infty} |\phi_{\tilde{f}}(t,x) - \phi_{\tilde{f}}(t,\omega)| = 0$. In V_δ take coordinates $(\tau, u) \in [-\delta,\delta] \times \tilde{W}^s_{\omega^*}$ with the local chart $(\tau, u) \to \phi_{\tilde{f}}(\tau, u)$. Let $\tilde{T}$ be the period, i.e. $\phi_{\tilde{f}}(\tilde{T}, \omega^*) = \omega^*$ and take a metric d_0 in $\tilde{W}^s_{\omega^*} \cap V_\delta$ and a $\tilde{\mu} < 1$ such that for any u, v there holds $d_0(\phi_{\tilde{f}}(\tilde{T},u), \phi_{\tilde{f}}(\tilde{T},v)) \le \tilde{\mu} d_0(u,v)$. In V_δ set $d(\phi_{\tilde{f}}(t,u), \phi_{\tilde{f}}(\tau,v)) := d_0(u,v) + |t-\tau|$. Let $T \ge \tilde{T}$ and $C > 0$ be such that for any $x \in K$ holds $\phi_{\tilde{f}}([0,T],x) \cap V_{\delta/3} \ne \emptyset$ and

$$|\phi_f(mh,x) - \psi_h^m(x)| \le Ch^p \qquad \text{for } 0 \le mh \le T \tag{3.5}$$

$$|\phi_f(t,x) - \phi_{\tilde{f}}(t,x)| \le Ch^p \qquad \text{for } 0 \le t \le T \quad . \tag{3.6}$$

Take $\tilde{x}_0 = x_0 \in K$ and $m_0 \le T/h$ s.t. $x_1 := \psi_h^{m_0}(x_0) \in V_{\delta/2}$. Let $x_{j+1} := \psi_h^{m_j}(x_j)$ be the next return of x_j to $V_{\delta/2}$ after one revolution. Set $\tilde{x}_{j+1} := \phi_{\tilde{f}}(m_j h, \tilde{x}_j)$. Let (u_j, τ_j) and $(\tilde{u}_j, \tilde{\tau}_j)$ be the coordinates of x_j and $\tilde{x}_j$, respectively. Since Γ^h is ψ_h– and $\phi_{\tilde{f}}$–invariant and $\psi_h^{m_j}$ is $O(h^p)$ $\mathcal{C}^1$–close to $\phi_{\tilde{f}}(m_j h, \cdot)$ we have for all $j \ge 1$:

$$\begin{aligned} d_0(u_{j+1}, \tilde{u}_{j+1}) &\le \mu d_0(u_j, \tilde{u}_j) \quad \text{and} \\ d_0(u_{j+1}, \omega^*) &\le \mu d_0(u_j, \omega^*) \end{aligned} \tag{3.7}$$

with some $\mu < 1$. Set $\varphi(m,t) := \tau(\psi_h^m(\omega(t)))$. By the definition of the τ–coordinate

$$\begin{aligned} \tau(\phi_{\tilde{f}}(m_j h, \tilde{x}_j)) &= \tau(\phi_{\tilde{f}}(m_j h, \omega(\tilde{\tau}_j))) = \tilde{\tau}_j + m_j h \mod \tilde{T} \qquad \text{and} \\ \tau(\psi_h^{m_j}(x_j)) &= \varphi(m_j, \tau_j) + O(h^p d_0(u_j, \omega^*)) \quad . \end{aligned} \tag{3.8}$$

Set $\hat{\omega}_1 := \omega(\tau_1)$ and let $\hat{\tau}_j$'s be the τ–coordinates of $\hat{\omega}_j$'s defined by $\hat{\omega}_{j+1} := \psi_h^{m_j}(\hat{\omega}_j)$. The following gives a bound fo $|\tau_j - \hat{\tau}_j|$

Lemma 3.5. $|\tau_j - \hat{\tau}_j| \le Ch$ *for all* $j \ge 1$.

Proof. Assume first $p \ge 2$. We show that for h small

$$\varphi(-1, \hat{\tau}_j) < \tau_j < \varphi(1, \hat{\tau}_j) \quad . \tag{3.9}$$

By (3.7) and (3.8) $|\tau_{j+1} - \varphi(m_j, \tau_j)| \le Ch^p \mu^j$. Set

$$\lambda := \min_{j,t} \partial_t \varphi(m_j, t) \ .$$

Then $\lambda = 1 - O(h^p)$. Take h small enough so that

$$\frac{Ch^p}{1-\mu/\lambda} < \alpha_1 := \varphi(1,\tau_1) - \tau_1 \quad . \tag{3.10}$$

Set $\alpha_j := \varphi(1,\hat{\tau}_j) - \tau_j$ and $\sigma = Ch^p$. We have $\alpha_1 > 0$ and if $\alpha_j > 0$ then

$$\begin{aligned}\alpha_{j+1} &\geq \varphi(m_j+1,\hat{\tau}_j) - \varphi(m_j,\tau_j) - \sigma\mu^j \\ &\geq \lambda[\varphi(1,\hat{\tau}_j) - \tau_j] - \sigma\mu^j = \lambda\alpha_j - \sigma\mu^j \quad ,\end{aligned}$$

which implies – as long as α_{j-1} stays nonnegative:

$$\alpha_j \geq \lambda^{j-1}\left(\alpha_1 - \sigma\sum_{i=1}^{j-1}\left(\frac{\mu}{\lambda}\right)^j\right) \quad .$$

By (3.10) α_j is positive for all j . The lower bound of (3.9) is similar.

If $p=1$ take m (independent of small h) such that

$$\sigma := \mu^m \frac{Ch}{1-\mu/\lambda} < \varphi(1,\tau_m) - \tau_m \quad .$$

The first m steps cause an mCh–error, and after those proceed as above. □

And a bound for $|\tilde{\tau}_j - \hat{\tau}_j|$ comes from

Claim 3.6. *For given $j > 0$ let n_j be the smallest integer such that $\rho n_j \geq j$. Then for any $\omega \in \Gamma^h$ $\psi_h^{n_j}(\omega)$ is between $\psi_h^{-1}(\omega)$ and $\psi_h(\omega)$ in the natural order on Γ^h .*

Proof. Set $J := [\omega, \psi_h(\omega))$. If $\psi_h^{n_j}(\omega) \geq \psi_h(\omega)$ then for the j:th return to J: $\psi_h^{k_1}(\omega) \in J$ holds $k_1 \leq n_j - 1$ and for the mj:th return $\psi_h^{k_m}(\omega) \in J$ holds $k_m \leq m(n_j-1)$. Thus (see [5])

$$\rho \geq \lim_{m\to\infty} \frac{mj}{m(n_j-1)} = \frac{j}{n_j-1}$$

a contradiction. The case $\psi_h^{n_j}(\omega) \leq \psi_h^{-1}(\omega)$ is similar. □

Since a similar claim holds for $\phi_{\tilde{f}}(h,\cdot)$ we get $|\tilde{\tau}_j - \hat{\tau}_j| \leq Ch$ and finally $|\tau_j - \tilde{\tau}_j| \leq Ch$. For the function τ of the theorem take $\tau(x_0, j)$ such that $\psi_h^j(x_0)$ and $\phi_{\tilde{f}}(\tau(x_0,j),x_0)$ are in the same $\tilde{W}_\omega^s$. We have the correspondences:

$$\sum_{i=0}^{j-1} m_j h \to \tilde{\tau}_j \quad \text{and} \quad \tau(x_0, \sum_{i=0}^{j-1} m_j) \to \tau_j \quad .$$

By the usual theory of numerical ODE's and checking the constructions above we can take the constant C to be continuous in x_0 and thus by compactness one for all x_0 in K .

This completes the proof for the case $k = 0$. Then use the induction:
The case $k+1$ follows from case k by applying it to the larger system:

$$\begin{aligned} \partial_t \phi(t,x) &= f(\phi(t,x)) \\ \partial_t \phi_x(t,x) &= f_x(\phi(t,x))\phi_x(t,x) \end{aligned} \tag{3.11}$$

using the assumed commutativity of discretation and differentiation as in [4]. □

Lemma 3.5 is probably not the best possible. Ch^p could be expected. This holds if the *lamination* (see [7]) (which is continuous) is Lipschitz–continuous. However, since Claim 3.6 cannot be improved in this generality (Example 2.1) we content ourselves with this weak one.

Remark 3.1. In the preliminary version of this paper presented at the IMACS world conference Dublin, 1991, the previous theorem was stated for hyperbolic periodic solutions. The proof for that case needs (in the construction of $\tilde{f}$) the $O(h^p)$ $\mathcal{C}^k$–closeness (i.e. Lipschitz–continuity) of the local stable and unstable manifolds as well. Since such a result is not found in the literature and the proof of it would be long, the more general case is omitted here. If, however, one is satisfied with just closeness of $\tilde{f}$ to f one can use the continuity of the stable and unstable manifolds ([7]) and apply the techniques above.

Remark 3.2. For an uncountable set of h's the rotation number of ψ_h on Γ^h is irrational. For those h's (see e.g. [5]) $\psi_{h|\Gamma^h}$ is topologically conjugate to rotation. This returns the possibility for $\tilde{P}(\infty, m, \infty, \infty, 0, 0, \infty)$ in this case. Here tilde denotes a weaker definition of P, where $(0, h_o)$ is replaced by the uncountable subset.

4. Discussion

We have not touched the third argument of P — the closeness of the initial point — yet. Is it possible that our numerical trajectory is all the time close to a true trajectory with just a different starting point? Such a shadowing property is true in certain hyperbolic situations of maps (see: Shadowing Lemma, e.g.[8]). If we want similar results here we have to consider again the drift along trajectories separately from other deviations as in the case of a periodic trajectory. This is because we cannot have hyperbolicity "in the direction of time." Anyway, this is also a very interesting subject.

References

1. D.K. Arrowsmith and C.M. Place, *Dynamical systems*, Cambridge University Press, Cambridge, 1990.
2. S-N. Chow and J. Hale, *Methods of bifurcation theory*, Springer-Verlag, New York, 1982.
3. Timo Eirola, *Invariant curves of one-step methods*, BIT **28** (1988), 113–122.
4. ______, *Two concepts for numerical periodic solutions of ode's*, Appl. Math. Comp. **31** (1989), 121–131.
5. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcation of vector fields*, Springer-Verlag, New York, 1983.
6. P. Hartman, *Ordinary differential equations*, John Wiley & Sons, New York–London–Sydney, 1964.

7. M.W. Hirsch, C.C. Pugh, and M. Shub, *Invariant manifolds*, Springer-Verlag, Berlin-Heidelberg, 1977.
8. M. Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987.
9. Andrew Stuart, *Private communication.*

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